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Interactions of a massless tensor field with the mixed symmetry of the Riemann tensor. No-go results

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Abstract. Non-trivial, consistent interactions of a free, massless tensor field $t_{\mu\nu|\alpha\beta}$ with the mixed symmetry of the Riemann tensor are studied in the following cases: self-couplings, cross-interactions with a Pauli–Fierz field and cross-couplings with purely matter theories. The main results, obtained from BRST cohomological techniques under the assumptions of smoothness, locality, Lorentz covariance and Poincaré invariance of the deformations, combined with the requirement that the interacting Lagrangian is at most second-order derivative, can be synthesized into the following results: no consistent self-couplings exist, but a cosmological-like term; no cross-interactions with the Pauli–Fierz field can be added; no non-trivial consistent cross-couplings with the matter theories such that the matter fields gain gauge transformations are allowed.

1 Introduction

Mixed symmetry type tensor fields [1–5] are involved in many physically interesting theories, like superstrings, supergravities or supersymmetric high spin theories. The study of gauge theories with mixed symmetry type tensor fields revealed several issues, like the dual formulation of field theories of spin two or higher [6–11], the impossibility of consistent cross-interactions in the dual formulation of linearized gravity [12] or a Lagrangian first-order approach [13,14] to some classes of free massless mixed symmetry type tensor gauge fields, suggestively resembling the tetrad formalism of general relativity. One of the most important aspects related to this type of gauge models is the analysis of their consistent interactions, among themselves, as well as with higher-spin gauge theories [15–19]. The best approach to this matter is the cohomological one, based on the deformation of the solution to the master equation [20]. The aim of our paper is to investigate the manifestly covariant consistent interactions involving a single, free, massless tensor gauge field $t_{\mu\nu|\alpha\beta}$, with the mixed symmetry of the Riemann tensor, in three distinct situations: self-couplings, interactions with the massless spin-two field (described in the free limit by the Pauli–Fierz action [21]), and couplings with purely matter theories.

Our procedure relies on the deformation of the solution to the master equation by means of local BRST cohomology. For each situation, we initially determine the associated free antifield-BRST symmetry s, which splits as the sum between the Koszul–Tate differential and the exterior longitudinal derivative only, $s = \delta + \gamma$. Then we solve the basic equations of the deformation procedure. Under the supplementary assumptions of smoothness, locality, Lorentz covariance and Poincaré invariance of the deformations, as well as of the maximum derivative order of the interacting Lagrangian being equal to two, we prove the following no-go results:
(i) the self-interactions of the tensor field with the mixed symmetry of the Riemann tensor do not modify either the original gauge algebra or the gauge transformations and, in fact, reduce to a cosmological-like term;
(ii) there are no consistent cross-interactions between such a tensor field and the Pauli–Fierz model. Only the Pauli–Fierz theory leads to consistent self-interactions, described by the Einstein–Hilbert action with a cosmological term, invariant under diffeomorphisms;
(iii) there are no couplings with purely matter theories such that the matter fields become endowed with gauge transformations.

This paper is organized in eight sections. Section 2 is dedicated to the Lagrangian formulation of the free massless tensor gauge field with mixed symmetry of the Riemann tensor, emphasizing its relationship with the generalized 3-differential complex. In Sect. 3 we construct the associated BRST symmetry and in Sect. 4 we briefly review the antifield-BRST deformation procedure. The following three sections represent the core of the paper and respectively address the problem of self-interactions, interactions with the Pauli–Fierz field, and couplings with purely matter fields. Section 8 ends the paper with the main conclusions.

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2 Free model

2.1 Field equations and gauge transformations

The starting point is given by the free Lagrangian action

$$
\begin{aligned}
&S_0[t_{\mu\nu|\alpha\beta}] \\
&= \int d^D x \left(\frac{1}{8} \left(\partial^\lambda t^{\mu\nu|\alpha\beta}\right)\left(\partial_\lambda t_{\mu\nu|\alpha\beta}\right) \right. \\
&\quad -\frac{1}{2}\left(\partial_\mu t^{\mu\nu|\alpha\beta}\right)\left(\partial^\lambda t_{\lambda\nu|\alpha\beta}\right) - \left(\partial_\mu t^{\mu\nu|\alpha\beta}\right)\left(\partial_\beta t_{\nu\alpha}\right) \\
&\quad -\frac{1}{2}\left(\partial^\lambda t^{\nu\beta}\right)\left(\partial_\lambda t_{\nu\beta}\right) + \left(\partial_\nu t^{\nu\beta}\right)\left(\partial^\lambda t_{\lambda\beta}\right) \\
&\quad \left. -\frac{1}{2}\left(\partial_\nu t^{\nu\beta}\right)\left(\partial_\beta t\right) + \frac{1}{8}\left(\partial^\lambda t\right)\left(\partial_\lambda t\right) \right), \qquad (1)
\end{aligned}
$$

in a Minkowski-flat spacetime of dimension $D \geq 5$, endowed with a metric tensor of "mostly plus" signature $\sigma_{\mu\nu} = \sigma^{\mu\nu} = (-++++\ldots)$. The massless tensor field $t_{\mu\nu|\alpha\beta}$ of degree four has the mixed symmetry of the linearized Riemann tensor, and hence transforms according to an irreducible representation of $GL(D,\mathbb{R})$, corresponding to a rectangular Young diagram $(2,2)$ with two columns and two rows, so it is separately antisymmetric in the pairs $\{\mu,\nu\}$ and $\{\alpha,\beta\}$, is symmetric under the interchange of these pairs ($\{\mu,\nu\} \longleftrightarrow \{\alpha,\beta\}$), $t_{\mu\nu|\alpha\beta} = t_{\alpha\beta|\mu\nu}$, and satisfies the identity

$$t_{[\mu\nu|\alpha]\beta} \equiv 0 \qquad (2)$$

associated with the above diagram, which we will refer to as the Bianchi I identity. Here and in the sequel the symbol $[\mu\nu\ldots]$ denotes the operation of antisymmetrization with respect to the indices between brackets, without normalization factors. (For instance, the left-hand side of (2) contains only the three terms $t_{[\mu\nu|\alpha]\beta} = t_{\mu\nu|\alpha\beta} + t_{\nu\alpha|\mu\beta} + t_{\alpha\mu|\nu\beta}$.) The notation $t_{\nu\beta}$ signifies the simple trace of the original tensor field, which is symmetric, $t_{\nu\beta} = \sigma^{\mu\alpha} t_{\mu\nu|\alpha\beta}$, while t denotes its double trace, which is a scalar, $t = \sigma^{\nu\beta} t_{\nu\beta}$. A generating set of gauge transformations for the action (1) reads

$$\delta_\epsilon t_{\mu\nu|\alpha\beta} = \partial_\mu \epsilon_{\alpha\beta|\nu} - \partial_\nu \epsilon_{\alpha\beta|\mu} + \partial_\alpha \epsilon_{\mu\nu|\beta} - \partial_\beta \epsilon_{\mu\nu|\alpha}, \qquad (3)$$

with the bosonic gauge parameters $\epsilon_{\mu\nu|\alpha}$ transforming according to an irreducible representation of $GL(D,\mathbb{R})$, corresponding to a Young diagram $(2,1)$ with two columns and two rows, being therefore antisymmetric in the pair $\{\mu,\nu\}$ and satisfying the identity

$$\epsilon_{[\mu\nu|\alpha]} \equiv 0. \qquad (4)$$

The identity (4) is required in order to ensure that the gauge transformations (3) obey the same Bianchi I identity as the fields themselves, namely, $\delta_\epsilon t_{[\mu\nu|\alpha]\beta} \equiv 0$. The above generating set of gauge transformations is abelian and off-shell first-stage reducible since if we make the transformation

$$\epsilon_{\mu\nu|\alpha} = 2\partial_\alpha \theta_{\mu\nu} - \partial_{[\mu}\theta_{\nu]\alpha}, \qquad (5)$$

with $\theta_{\mu\nu}$ an arbitrary antisymmetric tensor ($\theta_{\mu\nu} = -\theta_{\nu\mu}$), then the gauge transformations of the tensor field identically vanish, $\delta_\epsilon t_{\mu\nu|\alpha\beta} \equiv 0$. In the meantime, the transformation (5) agrees with the identity (4) obeyed by the gauge parameters.

The field equations resulting from the action (1) take the form

$$\frac{\delta S_0}{\delta t^{\mu\nu|\alpha\beta}} \equiv -\frac{1}{4} T_{\mu\nu|\alpha\beta} \approx 0, \qquad (6)$$

where

$$
\begin{aligned}
T_{\mu\nu|\alpha\beta} &= \Box t_{\mu\nu|\alpha\beta} \\
&+\partial^\rho \left(\partial_\mu t_{\alpha\beta|\nu\rho} - \partial_\nu t_{\alpha\beta|\mu\rho} + \partial_\alpha t_{\mu\nu|\beta\rho} - \partial_\beta t_{\mu\nu|\alpha\rho}\right) \\
&+ \left(\partial_\mu\partial_\alpha t_{\beta\nu} - \partial_\mu\partial_\beta t_{\alpha\nu} - \partial_\nu\partial_\alpha t_{\beta\mu} + \partial_\nu\partial_\beta t_{\alpha\mu}\right) \\
&-\frac{1}{2}\partial^\lambda\partial^\rho \left(\sigma_{\mu\alpha}\left(t_{\lambda\beta|\nu\rho} + t_{\lambda\nu|\beta\rho}\right) - \sigma_{\mu\beta}\left(t_{\lambda\alpha|\nu\rho} + t_{\lambda\nu|\alpha\rho}\right)\right. \\
&\left. - \sigma_{\nu\alpha}\left(t_{\lambda\beta|\mu\rho} + t_{\lambda\mu|\beta\rho}\right) + \sigma_{\nu\beta}\left(t_{\lambda\alpha|\mu\rho} + t_{\lambda\mu|\alpha\rho}\right)\right) \\
&-\Box\left(\sigma_{\mu\alpha}t_{\beta\nu} - \sigma_{\mu\beta}t_{\alpha\nu} - \sigma_{\nu\alpha}t_{\beta\mu} + \sigma_{\nu\beta}t_{\alpha\mu}\right) \\
&+\partial^\rho \left(\sigma_{\mu\alpha}\left(\partial_\beta t_{\nu\rho} + \partial_\nu t_{\beta\rho}\right) - \sigma_{\mu\beta}\left(\partial_\alpha t_{\nu\rho} + \partial_\nu t_{\alpha\rho}\right)\right. \\
&\left. - \sigma_{\nu\alpha}\left(\partial_\beta t_{\mu\rho} + \partial_\mu t_{\beta\rho}\right) + \sigma_{\nu\beta}\left(\partial_\alpha t_{\mu\rho} + \partial_\mu t_{\alpha\rho}\right)\right) \\
&-\frac{1}{2}\left(\sigma_{\mu\alpha}\partial_\beta\partial_\nu - \sigma_{\mu\beta}\partial_\alpha\partial_\nu - \sigma_{\nu\alpha}\partial_\beta\partial_\mu + \sigma_{\nu\beta}\partial_\alpha\partial_\mu\right) t \\
&- \left(\sigma_{\mu\alpha}\sigma_{\nu\beta} - \sigma_{\mu\beta}\sigma_{\nu\alpha}\right)\left(\partial^\lambda\partial^\rho t_{\lambda\rho} - \frac{1}{2}\Box t\right). \qquad (7)
\end{aligned}
$$

Obviously, the tensor $T_{\mu\nu|\alpha\beta}$ displays the same mixed symmetry properties as the tensor field $t_{\mu\nu|\alpha\beta}$. It is useful to compute its simple and double traces

$$
\begin{aligned}
&\sigma^{\mu\alpha} T_{\mu\nu|\alpha\beta} \\
&\equiv T_{\nu\beta} = (4-D)\left(\frac{1}{2}\partial^\lambda\partial^\rho\left(t_{\lambda\nu|\beta\rho} + t_{\lambda\beta|\nu\rho}\right)\right. \\
&\quad +\Box t_{\nu\beta} - \partial^\rho\left(\partial_\nu t_{\beta\rho} + \partial_\beta t_{\nu\rho}\right) \\
&\quad \left. +\frac{1}{2}\partial_\nu\partial_\beta t + \sigma_{\nu\beta}\left(\partial^\lambda\partial^\rho t_{\lambda\rho} - \frac{1}{2}\Box t\right)\right), \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
&\sigma^{\nu\beta} T_{\nu\beta} \\
&\equiv T = -(4-D)(3-D)\left(\partial^\lambda\partial^\rho t_{\lambda\rho} - \frac{1}{2}\Box t\right). \qquad (9)
\end{aligned}
$$

Obviously, its simple trace is a symmetric tensor, while its double trace is a scalar. The gauge invariance of the Lagrangian action (1) under the transformations (3) is equivalent to the fact that the functions defining the field equations are not all independent, but rather obey the Noether identities

$$\partial^\mu \frac{\delta S_0}{\delta t^{\mu\nu|\alpha\beta}} \equiv -\frac{1}{4}\partial^\mu T_{\mu\nu|\alpha\beta} = 0, \qquad (10)$$

while the first-stage reducibility shows that not all of the above Noether identities are independent. It can be checked that the functions (7) defining the field equations, the gauge generators, as well as the first-order reducibility functions, satisfy the general regularity assumptions from [22], such that the model under discussion is described by a normal gauge theory of Cauchy order equal to three.

2.2 Interpretation via the generalized 3-complex

This model describes a free gauge theory that can be interpreted in a consistent manner in terms of the generalized differential complex [23] $\Omega_2(\mathcal{M})$ of tensor fields with mixed symmetries corresponding to a maximal sequence of Young diagrams with two columns, defined on a pseudo-Riemannian manifold $\mathcal{M}$ of dimension D. Let us denote by $\bar{d}$ the associated operator (3-differential) that is third-order nilpotent, $\bar{d}^3 = 0$, and by $\Omega_2^p(\mathcal{M})$ the vector space spanned by the tensor fields from $\Omega_2(\mathcal{M})$ with p entries. The action of $\bar{d}$ on an element pertaining to $\Omega_2^p(\mathcal{M})$ results in a tensor from $\Omega_2^{p+1}(\mathcal{M})$ with one spacetime derivative, the action of $\bar{d}^2$ on a similar element leads to a tensor from $\Omega_2^{p+2}(\mathcal{M})$ containing two spacetime derivatives, while the action of $\bar{d}^3$ on any such element identically vanishes. In brief, the generalized 3-complex $\Omega_2(\mathcal{M})$ may suggestively be represented through the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & & & \cdots \\
 & & & & & & & \overset{\bar{d}^2}{\nearrow} & \\
 & & & & & & \Omega_2^8 & \overset{\bar{d}}{\to} & \cdots \\
 & & & & & \overset{\bar{d}^2}{\nearrow} & \uparrow^{\bar{d}} & & \\
 & & & & \boldsymbol{\Omega}_2^6 & \overset{\bar{d}}{\to} & \boldsymbol{\Omega}_2^7 & & \\
 & & & \overset{\bar{d}^2}{\nearrow} & \uparrow^{\bar{d}} & & & & \\
 & & \boldsymbol{\Omega}_2^4 & \overset{\bar{d}}{\to} & \boldsymbol{\Omega}_2^5 & & & & \\
 & \overset{\bar{d}^2}{\nearrow} & \uparrow^{\bar{d}} & & & & & & \\
\Omega_2^2 & \overset{\bar{d}}{\to} & \boldsymbol{\Omega}_2^3 & & & & & & \\
\overset{\bar{d}^2}{\nearrow} & \uparrow^{\bar{d}} & & & & & & & \\
\Omega_2^0 \; \overset{\bar{d}}{\to} & \Omega_2^1 & & & & & & &
\end{array}$$

where the third-order nilpotency of $\bar{d}$ means that any vertical arrow followed by the closest higher diagonal arrow maps to zero, and the same with respect to any diagonal arrow followed by the closest higher horizontal one. Its bold part emphasizes the sequences that apply to the model under discussion: the first one governs the dynamics and indicates the presence of some gauge symmetry

$$\begin{array}{ccccc}
\Omega_2^4 & & \Omega_2^6 & & \Omega_2^7 \\
\text{field} & \overset{\bar{d}^2}{\to} & \text{curvature} & \overset{\bar{d}}{\to} & \text{Bianchi II}, \\
t_{\mu\nu|\alpha\beta} & & F_{\mu\nu\lambda|\alpha\beta\gamma} & & \partial_{[\rho}F_{\mu\nu\lambda]|\alpha\beta\gamma} = 0
\end{array} \tag{11}$$

while the second sequence solves the gauge symmetry

$$\begin{array}{ccccc}
\Omega_2^3 & & \Omega_2^4 & & \Omega_2^6 \\
\text{gauge param.} & \overset{\bar{d}}{\to} & \text{gauge transf.} & \overset{\bar{d}^2}{\to} & \text{gauge inv. objects.} \\
\epsilon_{\alpha\beta|\mu} & & \delta_\epsilon t_{\mu\nu|\alpha\beta} & & \delta_\epsilon F_{\mu\nu\lambda|\alpha\beta\gamma} = 0
\end{array} \tag{12}$$

Let us discuss the previous sequences. Starting from the tensor field $t_{\mu\nu|\alpha\beta}$ from Ω_2^4, we can construct its curvature tensor $F_{\mu\nu\lambda|\alpha\beta\gamma}$, defined via

$$\begin{aligned}
&\left(\bar{d}^2 t\right)_{\mu\nu\lambda\alpha\beta\gamma} \sim \\
&F_{\mu\nu\lambda|\alpha\beta\gamma} = \partial_\lambda\partial_\gamma t_{\mu\nu|\alpha\beta} + \partial_\mu\partial_\gamma t_{\nu\lambda|\alpha\beta} + \partial_\nu\partial_\gamma t_{\lambda\mu|\alpha\beta} \\
&\quad + \partial_\lambda\partial_\alpha t_{\mu\nu|\beta\gamma} + \partial_\mu\partial_\alpha t_{\nu\lambda|\beta\gamma} + \partial_\nu\partial_\alpha t_{\lambda\mu|\beta\gamma} \\
&\quad + \partial_\lambda\partial_\beta t_{\mu\nu|\gamma\alpha} + \partial_\mu\partial_\beta t_{\nu\lambda|\gamma\alpha} + \partial_\nu\partial_\beta t_{\lambda\mu|\gamma\alpha},
\end{aligned} \tag{13}$$

which is second order in the spacetime derivatives and belongs to Ω_2^6. Thus, the curvature tensor transforms in an irreducible representation of $GL(D,\mathbb{R})$ and exhibits the symmetries of a rectangular two-column Young diagram $(3,3)$, being separately antisymmetric in the indices $\{\mu,\nu,\lambda\}$ and $\{\alpha,\beta,\gamma\}$, symmetric under the interchange $\{\mu,\nu,\lambda\} \longleftrightarrow \{\alpha,\beta,\gamma\}$, and obeying the (algebraic) Bianchi I identity

$$F_{[\mu\nu\lambda|\alpha]\beta\gamma} \equiv 0. \tag{14}$$

The action of $\bar{d}$ on $F_{\mu\nu\lambda|\alpha\beta\gamma}$ maps to zero,

$$\left(\bar{d}^3 t\right)_{\rho\mu\nu\lambda\alpha\beta\gamma} = \left(\bar{d}F\right)_{\rho\mu\nu\lambda\alpha\beta\gamma} \sim \partial_{[\rho}F_{\mu\nu\lambda]|\alpha\beta\gamma} \equiv 0, \tag{15}$$

and represents nothing but the (differential) Bianchi II identity for the curvature tensor. Since the curvature and its traces are the most general non-vanishing second-order derivative quantities in $\Omega_2(\mathcal{M})$ constructed from $t_{\mu\nu|\alpha\beta}$, we expect that the free field equations (6) completely rely on it. Equation (15) shows that the corresponding field equations cannot be all independent, but satisfy some Noether identities related to the Bianchi II identity of the curvature tensor. This already points out that the free Lagrangian action searched for must be invariant under a certain gauge symmetry. The second sequence, namely (12), gives the form of the gauge invariance. As the free field equations involve $F_{\mu\nu\lambda|\alpha\beta\gamma}$, it is natural to require that these are the most general gauge invariant quantities,

$$\delta_\epsilon \left(\bar{d}^2 t\right)_{\mu\nu\lambda\alpha\beta\gamma} \sim \delta_\epsilon F_{\mu\nu\lambda|\alpha\beta\gamma} = 0. \tag{16}$$

This matter is immediately solved if we take

$$\left(\bar{d}\epsilon\right)_{\mu\nu\alpha\beta} \sim \partial_\mu\epsilon_{\alpha\beta|\nu} - \partial_\nu\epsilon_{\alpha\beta|\mu} + \partial_\alpha\epsilon_{\mu\nu|\beta} - \partial_\beta\epsilon_{\mu\nu|\alpha} = \delta_\epsilon t_{\mu\nu|\alpha\beta}, \tag{17}$$

where the gauge parameters $\epsilon_{\mu\nu|\alpha}$ pertain to Ω_2^3, because, on account of the third-order nilpotency of $\bar{d}$, we find that

$$\delta_\epsilon F_{\mu\nu\lambda|\alpha\beta\gamma} \sim \left(\bar{d}^3\epsilon\right)_{\mu\nu\lambda\alpha\beta\gamma} \equiv 0. \tag{18}$$

Clearly, the relation (17) coincides with the gauge transformations (3).

We complete our discussion by exemplifying the construction of the free field equations. Let us denote by $S_0'[t_{\mu\nu|\alpha\beta}]$ a free, second-order derivative action that is gauge invariant under (17), and by $\delta S_0'/\delta t^{\mu\nu|\alpha\beta}$ its functional derivatives with respect to the fields, which are imposed to depend linearly on the undifferentiated curvature tensor. Then, as these functional derivatives must have the same mixed symmetry as $t_{\mu\nu|\alpha\beta}$, it follows that they necessarily determine a tensor from Ω_2^4. The operations that can be performed with respect to the curvature tensor in order to reduce its number of indices without increasing its derivative order is to take its simple, double, and respectively, triple traces,

$$F_{\mu\nu|\alpha\beta} = \sigma^{\lambda\gamma} F_{\mu\nu\lambda|\alpha\beta\gamma} \in \Omega_2^4, \tag{19}$$

$$F_{\mu\alpha} = \sigma^{\nu\beta} F_{\mu\nu|\alpha\beta} \in \Omega_2^2, \tag{20}$$

$$F = \sigma^{\mu\alpha} F_{\mu\alpha} \in \Omega_2^0, \tag{21}$$

where $F_{\mu\alpha}$ is symmetric and F is a scalar. The only combinations formed with these quantities that belong to Ω_2^4 are generated by

$$F_{\mu\nu|\alpha\beta}, \tag{22}$$

$$\sigma_{\mu\alpha} F_{\beta\nu} - \sigma_{\mu\beta} F_{\alpha\nu} - \sigma_{\nu\alpha} F_{\beta\mu} + \sigma_{\nu\beta} F_{\alpha\mu}, \tag{23}$$

and

$$\left(\sigma_{\mu\alpha}\sigma_{\nu\beta} - \sigma_{\mu\beta}\sigma_{\nu\alpha}\right) F, \tag{24}$$

so in principle $\delta S_0'/\delta t^{\mu\nu|\alpha\beta}$ can be written as a linear combination of (22)–(24) with coefficients that are real constants. However, the requirement that the above linear combination indeed stands for the functional derivatives of a sole functional restricts the parametrization of the functional derivatives, and therefore of the Lagrangian action, by means of one constant only,

$$\delta S_0'/\delta t^{\mu\nu|\alpha\beta} = \lambda \left(F_{\mu\nu|\alpha\beta} - \frac{1}{2}\left(\sigma_{\mu\alpha} F_{\beta\nu} - \sigma_{\mu\beta} F_{\alpha\nu} - \sigma_{\nu\alpha} F_{\beta\mu} + \sigma_{\nu\beta} F_{\alpha\mu}\right) + \frac{1}{6}\left(\sigma_{\mu\alpha}\sigma_{\nu\beta} - \sigma_{\mu\beta}\sigma_{\nu\alpha}\right) F \right). \tag{25}$$

If in (25) we take the particular value

$$\lambda = -\frac{1}{4}, \tag{26}$$

we recover the Lagrangian action (1) together with the field equations (6). This also allows us to identify the expression of $T_{\mu\nu|\alpha\beta}$ from (6) and (7) in terms of the curvature tensor like

$$T_{\mu\nu|\alpha\beta} = \left(F_{\mu\nu|\alpha\beta} - \frac{1}{2}\left(\sigma_{\mu\alpha} F_{\beta\nu} - \sigma_{\mu\beta} F_{\alpha\nu} - \sigma_{\nu\alpha} F_{\beta\mu} + \sigma_{\nu\beta} F_{\alpha\mu}\right) + \frac{1}{6}\left(\sigma_{\mu\alpha}\sigma_{\nu\beta} - \sigma_{\mu\beta}\sigma_{\nu\alpha}\right) F \right). \tag{27}$$

At this point, we can easily see the relationship of the field equations (6) and their Noether identities (10) with the curvature tensor (13) and accompanying Bianchi II identity (15). First, we observe that the field equations (6) are completely equivalent with the vanishing of the simple trace of the curvature tensor

$$T_{\mu\nu|\alpha\beta} \approx 0 \Longleftrightarrow F_{\mu\nu|\alpha\beta} \approx 0. \tag{28}$$

The direct statement holds due to the fact that $T_{\mu\nu|\alpha\beta}$ is expressed only through $F_{\mu\nu|\alpha\beta}$ and its traces, such that its vanishing implies $F_{\mu\nu|\alpha\beta} \approx 0$. The converse implication holds because the vanishing of the second and respectively third component in the right-hand side of (27) is a simple consequence of $F_{\mu\nu|\alpha\beta} \approx 0$. Second, the Noether identities (10) are a direct consequence of the Bianchi II identity for the curvature tensor,

$$\partial_{[\mu} F_{\alpha\beta\lambda]|\nu\rho\theta} \equiv 0 \Rightarrow \partial^\mu T_{\mu\nu|\alpha\beta} \equiv 0. \tag{29}$$

Indeed, on the one hand the relation (27) yields

$$\partial^\mu T_{\mu\nu|\alpha\beta} = \partial^\mu F_{\mu\nu|\alpha\beta} - \frac{1}{2}\partial_{[\alpha} F_{\beta]\nu} + \frac{1}{2}\sigma_{\nu[\alpha}\left(\partial^\mu F_{\beta]\mu} - \frac{1}{3}\partial_{\beta]} F\right). \tag{30}$$

On the other hand, simple computation leads to

$$\sigma^{\lambda\theta}\sigma^{\mu\rho}\partial_{[\mu} F_{\alpha\beta\lambda]|\nu\rho\theta} = -2\left(\partial^\mu F_{\mu\nu|\alpha\beta} - \frac{1}{2}\partial_{[\alpha} F_{\beta]\nu}\right), \tag{31}$$

$$2\sigma^{\nu\beta}\left(\partial^\mu F_{\mu\nu|\alpha\beta} - \frac{1}{2}\partial_{[\alpha} F_{\beta]\nu}\right) = 3\left(\partial^\mu F_{\alpha\mu} - \frac{1}{3}\partial_\alpha F\right). \tag{32}$$

Thus, according to (31) and (32) we can state that the Bianchi II identity for the curvature tensor implies identically vanishing of the right-hand side of (30), and hence enforces the Noether identities (10) for the action (1).

Next, we point out the relation between the generalized cohomology of the 3-complex $\Omega_2(\mathcal{M})$ and our model. The generalized cohomology of the 3-complex $\Omega_2(\mathcal{M})$ is given by the family of graded vector spaces $H_k(\bar{d}) = \mathrm{Ker}\left(\bar{d}^k\right)/\mathrm{Im}\left(\bar{d}^{3-k}\right)$, with $k = 1, 2$. Each vector space $H_k(\bar{d})$ splits into the cohomology spaces $H_{(k)}^p(\Omega_2(\mathcal{M}))$, defined as the equivalence classes of tensors from $\Omega_2^p(\mathcal{M})$ that are $\bar{d}^k$-closed, with any two such tensors that differ by a $\bar{d}^{3-k}$-exact element in the same equivalence class. The spaces $H_{(k)}^p$ are not empty in general, even if $\mathcal{M}$ has a trivial topology. However, in the case where $\mathcal{M}$ (assumed to be of dimension D) has the topology of $\mathbb{R}^D$, the generalized Poincaré lemma [23] applied to our situation states that the generalized cohomology of the 3-differential $\bar{d}$ on tensors represented by rectangular diagrams with two columns is empty in the space $\Omega_2(\mathbb{R}^D)$ of maximal two-column tensors, $H_{(k)}^{2n}\left(\Omega_2\left(\mathbb{R}^D\right)\right) = 0$, for $1 \leq n \leq D-1$ and $k = 1, 2$. In particular, for $n = 3$ and $k = 1$ we find that $H_{(1)}^6\left(\Omega_2\left(\mathbb{R}^D\right)\right) = 0$ and thus, if the tensor $F_{\mu\nu\lambda|\alpha\beta\gamma}$ with the mixed symmetry of the curvature tensor is $\bar{d}$-closed, then it is also $\bar{d}^2$-exact. To put it otherwise, if this tensor satisfies the Bianchi II identity $\partial_{[\rho} F_{\mu\nu\lambda]|\alpha\beta\gamma} \equiv 0$, then there exists an element $t_{\mu\nu|\alpha\beta}$ with the mixed symmetry $(2,2)$, with the help of which $F_{\mu\nu\lambda|\alpha\beta\gamma}$ can precisely be written like in (13).

Finally, we observe that the formula (27) relates the functions defining the free field equations (6) to the curvature tensor by a generalized Hodge duality. The generalized cohomology of $\bar{d}$ on $\Omega_2(\mathcal{M})$ when $\mathcal{M}$ has the trivial topology of $\mathbb{R}^D$ together with this type of generalized Hodge duality reveal many important features of the free model under study. For example, if $\bar{T}_{\mu\nu|\alpha\beta}$ is a covariant tensor field with the mixed symmetry $(2,2)$ and satisfies the equations

$$\partial^\mu \bar{T}_{\mu\nu|\alpha\beta} = 0, \tag{33}$$

then there exists a tensor $\bar{\Phi}_{\mu\nu\rho|\alpha\beta\gamma} \in \Omega_2(\mathbb{R}^D)$ with the mixed symmetry of the curvature tensor, in terms of which

$$\bar{T}_{\mu\nu|\alpha\beta} = \partial^\rho \partial^\gamma \bar{\Phi}_{\mu\nu\rho|\alpha\beta\gamma} + c\left(\sigma_{\mu\alpha}\sigma_{\nu\beta} - \sigma_{\mu\beta}\sigma_{\nu\alpha}\right), \tag{34}$$

with c an arbitrary real constant. It is easy to check the above statement in connection with the functions (7) that define the field equations for the model under consideration. Indeed, direct computation provides us with $c = 0$ and

$$T_{\mu\nu|\alpha\beta} = \frac{1}{2}\partial^\rho \partial^\gamma \Phi_{\mu\nu\rho|\alpha\beta\gamma}, \tag{35}$$

where

$$\begin{aligned}
&\Phi_{\mu\nu\rho|\alpha\beta\gamma} \\
&= \sigma_{\gamma[\rho}t_{\mu\nu]|\alpha\beta} + \sigma_{\alpha[\rho}t_{\mu\nu]|\beta\gamma} + \sigma_{\beta[\rho}t_{\mu\nu]|\gamma\alpha} + \sigma_{\rho[\gamma}t_{\alpha\beta]|\mu\nu} \\
&\quad + \sigma_{\mu[\gamma}t_{\alpha\beta]|\nu\rho} + \sigma_{\nu[\gamma}t_{\alpha\beta]|\rho\mu} \\
&\quad - 2\left(\sigma_{\gamma[\rho}\sigma_{\mu]\alpha}t_{\beta\nu} + \sigma_{\gamma[\mu}\sigma_{\nu]\alpha}t_{\beta\rho} + \sigma_{\gamma[\nu}\sigma_{\rho]\alpha}t_{\beta\mu}\right. \\
&\quad + \sigma_{\alpha[\rho}\sigma_{\mu]\beta}t_{\gamma\nu} + \sigma_{\alpha[\mu}\sigma_{\nu]\beta}t_{\gamma\rho} + \sigma_{\alpha[\nu}\sigma_{\rho]\beta}t_{\gamma\mu} \\
&\quad \left. + \sigma_{\beta[\rho}\sigma_{\mu]\gamma}t_{\alpha\nu} + \sigma_{\beta[\mu}\sigma_{\nu]\gamma}t_{\alpha\rho} + \sigma_{\beta[\nu}\sigma_{\rho]\gamma}t_{\alpha\mu}\right) \\
&\quad + \left(\sigma_{\gamma[\rho}\sigma_{\mu]\alpha}\sigma_{\beta\nu} + \sigma_{\gamma[\mu}\sigma_{\nu]\alpha}\sigma_{\beta\rho} + \sigma_{\gamma[\nu}\sigma_{\rho]\alpha}\sigma_{\beta\mu}\right)t,
\end{aligned} \tag{36}$$

so that the corresponding $\Phi_{\mu\nu\rho|\alpha\beta\gamma}$ indeed displays the mixed symmetry of the curvature tensor.

3 Free BRST symmetry

In agreement with the general setting of the antibracket–antifield formalism, the construction of the BRST symmetry for the free theory under consideration starts with the identification of the BRST algebra on which the BRST differential s acts. The generators of the BRST algebra are of two kinds: fields/ghosts and antifields. The ghost spectrum for the model under study comprises the fermionic ghosts $\eta_{\alpha\beta|\mu}$ associated with the gauge parameters $\epsilon_{\alpha\beta|\mu}$ from (3), as well as the bosonic ghosts for ghosts $C_{\mu\nu}$ due to the first-stage reducibility parameters $\theta_{\mu\nu}$ in (5). In order to make compatible the behavior of $\epsilon_{\alpha\beta|\mu}$ and $\theta_{\mu\nu}$ with that of the corresponding ghosts, we ask that $\eta_{\alpha\beta|\mu}$ satisfy the same properties like the gauge parameters,

$$\eta_{\mu\nu|\alpha} = -\eta_{\nu\mu|\alpha},\ \eta_{[\mu\nu|\alpha]} \equiv 0 \tag{37}$$

and that $C_{\mu\nu}$ is antisymmetric. The antifield spectrum is organized into the antifields $t^{*\mu\nu|\alpha\beta}$ of the original tensor field and those of the ghosts, $\eta^{*\mu\nu|\alpha}$ and $C^{*\mu\nu}$, of statistics opposite to that of the associated fields/ghosts. It is understood that $t^{*\mu\nu|\alpha\beta}$ is subject to some conditions similar to those satisfied by the tensor field

$$t^{*\mu\nu|\alpha\beta} = -t^{*\nu\mu|\alpha\beta} = -t^{*\mu\nu|\beta\alpha} = t^{*\alpha\beta|\mu\nu},\ t^{*[\mu\nu|\alpha]\beta} \equiv 0, \tag{38}$$

and, along the same lines, it is required that

$$\eta^{*\mu\nu|\alpha} = -\eta^{*\nu\mu|\alpha},\ \eta^{*[\mu\nu|\alpha]} \equiv 0,\ C^{*\mu\nu} = -C^{*\nu\mu}. \tag{39}$$

We will denote the simple and double traces of $t^{*\mu\nu|\alpha\beta}$ by

$$t^{*\nu\beta} = \sigma_{\mu\alpha}t^{*\mu\nu|\alpha\beta},\ t^{*\nu\beta} = t^{*\beta\nu},\ t^{*} = \sigma_{\nu\beta}t^{*\nu\beta}. \tag{40}$$

As both the gauge generators and reducibility functions for this model are field-independent, it follows that the associated BRST differential ($s^2 = 0$) splits into

$$s = \delta + \gamma, \tag{41}$$

where δ represents the Koszul–Tate differential ($\delta^2 = 0$), graded by the antighost number agh ($\mathrm{agh}(\delta) = -1$), while γ stands for the exterior derivative along the gauge orbits and turns out to be a true differential ($\gamma^2 = 0$) that anticommutes with δ ($\delta\gamma + \gamma\delta = 0$), whose degree is named pure ghost number pgh ($\mathrm{pgh}(\gamma) = 1$). These two degrees do not interfere ($\mathrm{agh}(\gamma) = 0$, $\mathrm{pgh}(\delta) = 0$). The overall degree that grades the BRST differential is known as the ghost number (gh) and is defined as the difference between the pure ghost number and the antighost number, such that $\mathrm{gh}(s) = \mathrm{gh}(\delta) = \mathrm{gh}(\gamma) = 1$. According to the standard rules of the BRST method, the corresponding degrees of the generators from the BRST complex are valued as

$$\mathrm{pgh}\left(t_{\mu\nu|\alpha\beta}\right) = 0,\ \mathrm{pgh}\left(\eta_{\mu\nu|\alpha}\right) = 1,\ \mathrm{pgh}\left(C_{\mu\nu}\right) = 2, \tag{42}$$

$$\mathrm{pgh}\left(t^{*\mu\nu|\alpha\beta}\right) = \mathrm{pgh}\left(\eta^{*\mu\nu|\alpha}\right) = \mathrm{pgh}\left(C^{*\mu\nu}\right) = 0, \tag{43}$$

$$\mathrm{agh}\left(t_{\mu\nu|\alpha\beta}\right) = \mathrm{agh}\left(\eta_{\mu\nu|\alpha}\right) = \mathrm{agh}\left(C_{\mu\nu}\right) = 0, \tag{44}$$

$$\mathrm{agh}\left(t^{*\mu\nu|\alpha\beta}\right) = 1,\ \mathrm{agh}\left(\eta^{*\mu\nu|\alpha}\right) = 2,\ \mathrm{agh}\left(C^{*\mu\nu}\right) = 3, \tag{45}$$

and the actions of δ and γ on them are given by

$$\gamma t_{\mu\nu|\alpha\beta} = \partial_\mu \eta_{\alpha\beta|\nu} - \partial_\nu \eta_{\alpha\beta|\mu} + \partial_\alpha \eta_{\mu\nu|\beta} - \partial_\beta \eta_{\mu\nu|\alpha}, \tag{46}$$

$$\gamma \eta_{\mu\nu|\alpha} = 2\partial_\alpha C_{\mu\nu} - \partial_{[\mu}C_{\nu]\alpha},\ \gamma C_{\mu\nu} = 0, \tag{47}$$

$$\gamma t^{*\mu\nu|\alpha\beta} = 0,\ \gamma \eta^{*\mu\nu|\alpha} = 0,\ \gamma C^{*\mu\nu} = 0, \tag{48}$$

$$\delta t_{\mu\nu|\alpha\beta} = 0,\ \delta \eta_{\mu\nu|\alpha} = 0,\ \delta C_{\mu\nu} = 0, \tag{49}$$

$$\delta t^{*\mu\nu|\alpha\beta} = \frac{1}{4}T^{\mu\nu|\alpha\beta},\ \delta \eta^{*\alpha\beta|\nu} = -4\partial_\mu t^{*\mu\nu|\alpha\beta},$$

$$\delta C^{*\mu\nu} = 3\partial_\alpha \eta^{*\mu\nu|\alpha}, \tag{50}$$

with $T_{\mu\nu|\alpha\beta}$ expressed in (7) and both δ and γ taken to act like right derivations.

The antifield-BRST differential is known to admit a canonical action in a structure named antibracket and defined by decreeing the fields/ghosts conjugated with the corresponding antifields, $s\cdot = (\cdot, S)$, where $(,)$ signifies the antibracket and S denotes the canonical generator of the BRST symmetry. It is a bosonic functional of ghost number zero involving both the field/ghost and antifield spectra, which obeys the classical master equation

$$(S, S) = 0. \tag{51}$$

The classical master equation is equivalent with the second-order nilpotency of s, $s^2 = 0$, while its solution encodes the entire gauge structure of the associated theory. Taking into account (46)–(50), as well as the actions of δ and γ in canonical form, we find that the complete solution to the master equation for the model under study reads

$$\begin{aligned} S = {} & S_0[t_{\mu\nu|\alpha\beta}] \\ & + \int d^D x \\ & \times \Big(t^{*\mu\nu|\alpha\beta} \left(\partial_\mu \eta_{\alpha\beta|\nu} - \partial_\nu \eta_{\alpha\beta|\mu} + \partial_\alpha \eta_{\mu\nu|\beta} - \partial_\beta \eta_{\mu\nu|\alpha} \right) \\ & + \eta^{*\mu\nu|\alpha} \left(2\partial_\alpha C_{\mu\nu} - \partial_{[\mu} C_{\nu]\alpha} \right) \Big). \end{aligned} \tag{52}$$

The main ingredients of the antifield-BRST symmetry derived in this section will be useful in the sequel at the analysis of consistent interactions that can be added to the action (1) without changing its number of independent gauge symmetries.

4 Brief review of the antifield-BRST deformation procedure

There are three main types of consistent interactions that can be added to a given gauge theory:
(i) the first type deforms only the Lagrangian action, but not its gauge transformations,
(ii) the second kind modifies both the action and its transformations, but not the gauge algebra, and
(iii) the third, and certainly most interesting category, changes everything, namely, the action, its gauge symmetries and the accompanying algebra.

The reformulation of the problem of consistent deformations of a given action and of its gauge symmetries in the antifield-BRST setting is based on the observation that if a deformation of the classical theory can be consistently constructed, then the solution to the master equation for the initial theory can be deformed into

$$\bar{S} = S + gS_1 + g^2 S_2 + O\left(g^3\right), \ \varepsilon\left(\bar{S}\right) = 0, \ \mathrm{gh}\left(\bar{S}\right) = 0, \tag{53}$$

such that

$$\left(\bar{S}, \bar{S}\right) = 0. \tag{54}$$

Here and in the sequel $\varepsilon(F)$ denotes the Grassmann parity of F. The projection of (54) on the various powers in the coupling constant induces the following tower of equations:

$$g^0 : (S, S) = 0, \tag{55}$$

$$g^1 : (S_1, S) = 0, \tag{56}$$

$$g^2 : \frac{1}{2}(S_1, S_1) + (S_2, S) = 0, \tag{57}$$

$$\vdots$$

The first equation is satisfied by hypothesis. The second one governs the first-order deformation of the solution to the master equation (S_1) and it shows that S_1 is a BRST co-cycle, $sS_1 = 0$, and hence it exists and is local. The remaining equations are responsible for the higher-order deformations of the solution to the master equation. No obstructions arise in finding solutions to them as long as no further restrictions, such as spacetime locality, are imposed. Obviously, only non-trivial first-order deformations should be considered, since trivial ones ($S_1 = sB$) lead to trivial deformations of the initial theory and can be eliminated by convenient redefinitions of the fields. Ignoring the trivial deformations, it follows that S_1 is a non-trivial BRST-observable, $S_1 \in H^0(s)$. Once the deformation equations (56)–(57), etc., have been solved by means of specific cohomological techniques, from the consistent non-trivial deformed solution to the master equation we can extract all the information on the gauge structure of the accompanying interacting theory.

5 Self-interactions

The first task of our paper is to study the consistent interactions that can be added to the free action (1) by means of solving the main deformation equations, namely, (56)–(57), etc. For obvious reasons, we consider only smooth, local, Lorentz-covariant and Poincaré-invariant deformations. If we choose the notation $S_1 = \int d^D x\, a$, with a a local function, then the local form of (56), which we have seen to control the first-order deformation of the solution to the master equation, becomes

$$sa = \partial_\mu m^\mu, \ \mathrm{gh}(a) = 0, \ \varepsilon(a) = 0, \tag{58}$$

for some m^μ, and it shows that the non-integrated density of the first-order deformation pertains to the local cohomology of s at ghost number zero, $a \in H^0(s|d)$, where d denotes the exterior spacetime differential. In order to analyze the above equation, we develop a according to the antighost number

$$a = \sum_{k=0}^{I} a_k, \ \mathrm{agh}(a_k) = k, \ \mathrm{gh}(a_k) = 0, \ \varepsilon(a_k) = 0, \tag{59}$$

and assume, without loss of generality, that a stops at some finite value I of the antighost number.[1] By taking into ac-

[1] This can be shown, for instance, like in [26] (Sect. 3), under the sole assumption that the interacting Lagrangian at the first order in the coupling constant, a_0, has a finite, but otherwise arbitrary derivative order.

count the decomposition (41) of the BRST differential, (58) is equivalent to a tower of local equations, corresponding to the various decreasing values of the antighost number

$$\gamma a_I = \partial_\mu \overset{(I)^\mu}{m}, \tag{60}$$

$$\delta a_I + \gamma a_{I-1} = \partial_\mu \overset{(I-1)^\mu}{m}, \tag{61}$$

$$\delta a_k + \gamma a_{k-1} = \partial_\mu \overset{(k-1)^\mu}{m},\ I-1 \geq k \geq 1, \tag{62}$$

where $\left(\overset{(k)^\mu}{m}\right)_{k=\overline{0,I}}$ are some local currents, with $\mathrm{agh}\left(\overset{(k)^\mu}{m}\right) = k$. It can be proved[2] that one can replace (60) at strictly positive antighost numbers with

$$\gamma a_I = 0,\ I > 0. \tag{63}$$

In conclusion, under the assumption that $I > 0$, the representative of highest antighost number from the non-integrated density of the first-order deformation can always be taken to be γ-closed, such that (58) associated with the local form of the first-order deformation is completely equivalent to the tower of equations (63), and (61) and (62).

Before proceeding to the analysis of the solutions to the first-order deformation equations, we briefly comment on the uniqueness and triviality of such solutions. Due to the second-order nilpotency of γ ($\gamma^2 = 0$), the solution to the top equation (63) is clearly unique up to γ-exact contributions,

$$a_I \rightarrow a_I + \gamma b_I,\ \mathrm{agh}\,(b_I) = I,\ \mathrm{pgh}\,(b_I) = I-1,\ \varepsilon\,(b_I) = 1. \tag{64}$$

Meanwhile, if it turns out that a_I reduces to γ-exact terms only, $a_I = \gamma b_I$, then it can be made to vanish, $a_I = 0$. In other words, the non-triviality of the first-order deformation a is translated at its highest antighost number component into the requirement that

$$a_I \in H^I(\gamma), \tag{65}$$

where $H^I(\gamma)$ denotes the cohomology of the exterior longitudinal derivative γ at pure ghost number equal to I. At the same time, the general condition on the non-integrated density of the first-order deformation to be in a non-trivial cohomological class of $H^0(s|d)$ shows on the one hand that the solution to (58) is unique up to s-exact pieces plus total divergences:

$$a \rightarrow a + sb + \partial_\mu n^\mu, \tag{66}$$

$$\mathrm{gh}\,(b) = -1,\ \varepsilon\,(b) = 1,\ \mathrm{gh}\,(n^\mu) = 0,\ \varepsilon\,(n^\mu) = 0,$$

and on the other hand that if the general solution to (58) is found to be completely trivial, $a = sb + \partial_\mu n^\mu$, then it can be made to vanish, $a = 0$.

In the light of the above discussion, we pass to the investigation of the solutions to (63), and (61) and (62). We have seen that a_I belongs to the cohomology of the exterior longitudinal derivative (see (65)), such that we need to compute $H(\gamma)$ in order to construct the component of highest antighost number from the first-order deformation. This matter is solved with the help of the definitions (46)–(48).

[2] The proof is given in Corollary 3.1 from [27].

5.1 $H(\gamma)$ and $H(\delta|d)$

The formula (48) shows that all the antifields

$$\chi^{*\Delta} = \left(t^{*\mu\nu|\alpha\beta}, \eta^{*\mu\nu|\alpha}, C^{*\mu\nu}\right) \tag{67}$$

belong (non-trivially) to $H^0(\gamma)$. From the definition (46) and recalling the general discussion from Sect. 2 on the relationship between the model under investigation and the 3-differential complex, we infer that the most general γ-closed (and obviously non-trivial) elements constructed in terms of the original tensor field are the components of the curvature tensor (13) and their spacetime derivatives, so all these pertain to $H^0(\gamma)$.

Using the first definition in (47), we notice that there is no γ-closed linear combination of the undifferentiated ghosts of pure ghost number one. On behalf of the same definition, we investigate the existence of γ-closed linear combinations in the first-order derivatives of these ghosts. By direct computation, it is easy to see that the most general γ-closed quantities in the first-order derivatives of the pure ghost number one ghosts have the mixed symmetry of the tensor field $t_{\mu\nu|\alpha\beta}$ itself

$$M_{\mu\nu|\alpha\beta} = \partial_\mu \eta_{\alpha\beta|\nu} - \partial_\nu \eta_{\alpha\beta|\mu} + \partial_\alpha \eta_{\mu\nu|\beta} - \partial_\beta \eta_{\mu\nu|\alpha}. \tag{68}$$

However, with the help of (46) it is obvious that $M_{\mu\nu|\alpha\beta}$ is γ-exact, $M_{\mu\nu|\alpha\beta} = \gamma t_{\mu\nu|\alpha\beta}$, and thus it must be discarded from $H^1(\gamma)$ as being trivial. Along the same line, one can prove that the only γ-closed combinations with $N \geq 2$ spacetime derivatives of the ghosts $\eta_{\mu\nu|\alpha}$ are actually polynomials with $(N-1)$ derivatives in the elements $M_{\mu\nu|\alpha\beta}$, so they are γ-exact, and hence trivial in $H^1(\gamma)$. In conclusion, there is no non-trivial object constructed out of the ghosts $\eta_{\mu\nu|\alpha}$ and their derivatives in $H^1(\gamma)$, which implies that $H^1(\gamma) = 0$ as there are no other ghosts of pure ghost number equal to one in the BRST complex. The BRST complex for the model under consideration contains no other ghosts with odd pure ghost numbers, so we conclude that

$$H^{2l+1}(\gamma) = 0,\ \text{for all}\ l \geq 0. \tag{69}$$

The definitions (47) show that the undifferentiated ghosts of pure ghost number equal to two, $C_{\mu\nu}$, belong to $H(\gamma)$. The γ-closedness of $C_{\mu\nu}$ further implies that all their derivatives are also γ-closed. Let us see which of these derivatives are trivial. Regarding their first-order derivatives, from the first relation in (47) we observe that their symmetric part is γ-exact

$$\partial_{(\mu} C_{\nu)\alpha} \equiv \gamma\left(-\frac{1}{3}\eta_{\alpha(\mu|\nu)}\right), \tag{70}$$

where $(\mu\nu\ldots)$ denotes plain symmetrization with respect to the indices between brackets without normalization factors, such that $\partial_{(\mu}C_{\nu)\alpha}$ will be removed from $H(\gamma)$. Meanwhile, their antisymmetric part $\partial_{[\mu}C_{\nu]\alpha}$ is not γ-exact, and hence can be taken as a non-trivial representative of $H(\gamma)$. After some calculations, we find that all the second-order derivatives of the ghosts for ghosts are γ-exact:

$$\partial_\alpha\partial_\beta C_{\mu\nu} = \frac{1}{12}\gamma\left(3\left(\partial_\alpha\eta_{\mu\nu|\beta} + \partial_\beta\eta_{\mu\nu|\alpha}\right) + \partial_{[\mu}\eta_{\nu]\,(\alpha|\beta)}\right), \tag{71}$$

and so will be their higher-order derivatives. In conclusion, the only non-trivial combinations in $H(\gamma)$ constructed from the ghosts of pure ghost number equal to two are polynomials in $C_{\mu\nu}$ and $\partial_{[\mu}C_{\nu]\alpha}$. Combining this result with the previous one on $H^0(\gamma)$ being non-vanishing, we have actually proved that only the even cohomological spaces of the exterior longitudinal derivative, $H^{2l}(\gamma)$ with $l \geq 0$, are non-vanishing.

Under these circumstances, it follows that (63) possesses non-trivial solutions only for $I = 2J$, where the general form of a_{2J} for $J > 0$ is (up to irrelevant, γ-exact contributions)

$$a_I \equiv a_{2J} = \alpha_{2J}\left([\chi^{*\Delta}], [F_{\mu\nu\lambda|\alpha\beta\gamma}]\right) \mathrm{e}^{2J}(C_{\mu\nu}, \partial_{[\mu}C_{\nu]\alpha}), \quad J > 0, \tag{72}$$

where the notation $f([q])$ means that f depends on q and its spacetime derivatives up to a finite order. The coefficients α_{2J} are γ-invariant:

$$\gamma\alpha_{2J} = 0, \tag{73}$$

and exhibit the properties $\varepsilon(\alpha_{2J}) = 0$, $\mathrm{pgh}(\alpha_{2J}) = 0$ and $\mathrm{agh}(\alpha_{2J}) = 2J$, while the symbol e^{2J} stands for a generic notation of the elements with pure ghost number equal to $2J$ of a basis in the space of polynomials in $C_{\mu\nu}$ and $\partial_{[\mu}C_{\nu]\alpha}$. The objects α_{2J} (obviously non-trivial in $H^0(\gamma)$) were taken to have a bounded number of derivatives, and therefore they are polynomials in the antifields $\chi^{*\Delta}$, in the curvature tensor $F_{\mu\nu\lambda|\alpha\beta\gamma}$, as well as in their derivatives. Due to their γ-closedness, they are called invariant polynomials. At zero antighost number, the invariant polynomials are polynomials in the curvature tensor $F_{\mu\nu\lambda|\alpha\beta\gamma}$ and its derivatives. The result that we can replace (60) with the less obvious one (63) is a nice consequence of the fact that the cohomology of the exterior spacetime differential is trivial in the space of invariant polynomials at strictly positive antighost numbers. This means that if the invariant polynomial α_I of strictly positive antighost number is annihilated by d, then it can be written like the d-variation of precisely an invariant polynomial. For details, see Sect. 3 in [27].

Replacing the solution (72) in (61) for $I = 2J$ and taking into account the definitions (47), we remark that a necessary (but not sufficient) condition for the existence of (non-trivial) solutions a_{2J-1} is that the invariant polynomials α_{2J} from (72) are (non-trivial) objects from the local cohomology of the Koszul–Tate differential $H(\delta|d)$ at antighost number $2J > 0$ and pure ghost number equal to zero[3], $\alpha_{2J} \in H_{2J}(\delta|d)$, i.e.

$$\delta\alpha_{2J} = \partial_\mu j^\mu,\ \varepsilon(j^\mu) = 1,\ \mathrm{agh}(j^\mu) = 2J-1,\ \mathrm{pgh}(j^\mu) = 0. \tag{74}$$

Consequently, we need to investigate some of the main properties of the local cohomology of the Koszul–Tate differential at pure ghost number zero and strictly positive antighost numbers in order to completely determine the component a_{2J} of highest antighost number in the first-order deformation. As we have discussed in Sect. 2, the free model under study is a normal gauge theory of Cauchy order equal to three. Using the general results from [24] (also see [12] and [25,26]), one can state that the local cohomology of the Koszul–Tate differential at pure ghost number zero is trivial at antighost numbers strictly greater than its Cauchy order

$$H_k(\delta|d) = 0,\ k > 3. \tag{75}$$

Moreover, if the invariant polynomial α_k, with $\mathrm{agh}(\alpha_k) = k \geq 3$, is trivial in $H_k(\delta|d)$, then it can be taken to be trivial also in $H_k^{\mathrm{inv}}(\delta|d)$

$$\left(\alpha_k = \delta b_{k+1} + \partial_\mu \overset{(k)}{c}{}^\mu,\ \mathrm{agh}(\alpha_k) = k \geq 3\right) \Rightarrow$$
$$\alpha_k = \delta\beta_{k+1} + \partial_\mu \overset{(k)}{\gamma}{}^\mu, \tag{76}$$

where β_{k+1} and $\overset{(k)}{\gamma}{}^\mu$ are invariant polynomials. [An element of $H_k^{\mathrm{inv}}(\delta|d)$ is defined via an equation similar to (74) for $2J \to k$, but with the corresponding current an invariant polynomial.] The result (76) is proved in Theorem 4.1 from [27]. It is important since it together with (75) ensures that all the local cohomology of the Koszul–Tate differential in the space of invariant polynomials is trivial in antighost numbers strictly greater than three,

$$H_k^{\mathrm{inv}}(\delta|d) = 0,\ k > 3. \tag{77}$$

Using the definitions (50), we can organize the non-trivial representatives of $\left(H_k(\delta|d)\right)_{k\geq 2}$ (at pure ghost number equal to zero) and $\left(H_k^{\mathrm{inv}}(\delta|d)\right)_{k\geq 2}$ as

agh	non – trivial representatives spanning $H_k(\delta\vert d)$ and $H_k^{\mathrm{inv}}(\delta\vert d)$
$k > 3$	none
$k = 3$	$C^{*\mu\nu}$
$k = 2$	$\eta^{*\mu\nu\vert\alpha}$

(78)

With the help of the above representatives we can construct in principle other non-trivial elements from $H(\delta|d)$ and $H^{\mathrm{inv}}(\delta|d)$ at strictly positive antighost numbers, which explicitly depend on the spacetime co-ordinates. For instance, the object $\eta^*_{\mu\nu|\alpha}f^{\mu\nu}x^\alpha$, with $f^{\mu\nu}$ some antisymmetric constants, belongs to both $H_2(\delta|d)$ and $H_2^{\mathrm{inv}}(\delta|d)$. However,

[3] We recall that the local cohomology $H(\delta|d)$ is completely trivial at both strictly positive antighost *and* pure ghost numbers (for instance, see [24], Theorem 5.4 and [28]).

we will discard such elements during the deformation procedure, since they would break the Poincaré invariance of the interactions. In contrast to the groups $(H_k(\delta|d))_{k\geq 2}$ and $\left(H_k^{\rm inv}(\delta|d)\right)_{k\geq 2}$, which are finite-dimensional, the cohomology $H_1(\delta|d)$ at pure ghost number zero, that is related to global symmetries and ordinary conservation laws, is infinite-dimensional since the theory is free. Fortunately, it will not be needed in the sequel.

The above results on $H(\delta|d)$ and $H^{\rm inv}(\delta|d)$ in strictly positive antighost number are important because they control the obstructions to removing the antifields from the first-order deformation. Indeed, due to (77) and (69) we can successively eliminate all the pieces of antighost number strictly greater than two from the non-integrated density of the first-order deformation by adding only trivial terms (for details, see Sect. 5 from [27]), so we can take, without loss of non-trivial objects, the condition

$$0 \leq I = 2J \leq 2 \tag{79}$$

in the development (59), which leaves us with a single eligible, strictly positive value, $I = 2J = 2$.

5.2 The case $I = 2$

Thus, for $I = 2J = 2$ we finally obtain that the expansion (59) becomes

$$a = a_0 + a_1 + a_2, \tag{80}$$

where its last component is written (up to γ-exact objects) in the form

$$a_2 = \alpha_2\left([t^{*\mu\nu|\alpha\beta}], [\eta^{*\mu\nu|\alpha}], [F_{\mu\nu\lambda|\alpha\beta\gamma}]\right) e^2\left(C_{\mu\nu}, \partial_{[\mu}C_{\nu]\alpha}\right), \tag{81}$$

with the elements of pure ghost number two spanned by

$$\left(C_{\mu\nu}, \partial_{[\mu}C_{\nu]\alpha}\right). \tag{82}$$

Taking into account the result from (78) at $k = 2$, we get

$$a_2 = \eta^*_{\mu\nu|\alpha}\left(f^{\mu\nu\alpha\beta\gamma}C_{\beta\gamma} + \bar{f}^{\mu\nu\alpha\beta\gamma\lambda}\partial_{[\beta}C_{\gamma]\lambda}\right), \tag{83}$$

where $f^{\mu\nu\alpha\beta\gamma}$ and $\bar{f}^{\mu\nu\alpha\beta\gamma\lambda}$ must be non-derivative constants. In the meantime, $f^{\mu\nu\alpha\beta\gamma}$ and $\bar{f}^{\mu\nu\alpha\beta\gamma\lambda}$ cannot be antisymmetric in all indices $\{\mu, \nu, \alpha\}$ (because in this event the identity $\eta^{*[\mu\nu|\alpha]} \equiv 0$ maps the corresponding terms to zero), which eventually leaves one candidate for a_2:

$$a_2 = c\eta^{*\mu\nu|\alpha}\partial_{[\mu}C_{\nu]\alpha}, \tag{84}$$

with c an arbitrary real constant. However, this term is easily shown to be trivial (γ-exact) on account of the first definition in (47) and of the identity $\eta^{*[\mu\nu|\alpha]} \equiv 0$, which allows us to add to a_2 any quantity proportional with $\eta^{*\mu\nu|\alpha}\partial_{[\mu}C_{\nu\alpha]}$ since it vanishes identically,

$$c\eta^{*\mu\nu|\alpha}\partial_{[\mu}C_{\nu]\alpha} = c\eta^{*\mu\nu|\alpha}\left(\partial_{[\mu}C_{\nu]\alpha} - \frac{2}{3}\partial_{[\alpha}C_{\mu\nu]}\right) = \gamma\left(-\frac{c}{3}\eta^{*\mu\nu|\alpha}\eta_{\mu\nu|\alpha}\right), \tag{85}$$

and so it can be discarded from (84) by setting

$$c = 0. \tag{86}$$

So far we have shown that there is no non-trivial a_2 in the right-hand side of (80),

$$a_2 = 0. \tag{87}$$

It is worth noticing that at this stage we have not used any a priori restriction on the number of derivatives from a_2, except that it is finite, but only the general requirements of smooth, local, Lorentz-covariant and Poincaré-invariant deformations. The assumption that the interactions contain at most two derivatives will only be needed below.

5.3 The case $I = 0$

Consequently, we pass to the next value of the maximum antighost number in the expansion (59), which, according to the restriction (79), excludes the value $I = 1$. Thus, we are only left with the possibility that the non-integrated density of the first-order deformation reduces to its antighost number zero component, which is nothing but the deformed Lagrangian at the first order in the coupling constant

$$a = a_0\left([t_{\mu\nu|\alpha\beta}]\right), \tag{88}$$

which must obey the equation

$$\gamma a_0 = \partial_\mu m^\mu. \tag{89}$$

There are two main types of solutions to the last equation. The first one corresponds to $m^\mu = 0$ and is given by functions in the field $t_{\mu\nu|\alpha\beta}$ and its derivatives that are invariant under the gauge transformations (3). As the components of the curvature tensor are the most general gauge invariant objects, it follows that

$$\gamma a'_0 = 0 \Rightarrow a'_0 = a'_0\left([F_{\mu\nu\lambda|\alpha\beta\gamma}]\right). \tag{90}$$

At this point we demand that the deformed gauge theory preserves the Cauchy order of the uncoupled model, which enforces the requirement that the interacting Lagrangian is of maximum order equal to two in the spacetime derivatives of the tensor field $t_{\mu\nu|\alpha\beta}$ at each order in the coupling constant. In turn, this requirement leads to $a'_0 = 0$ (we have excluded the solutions linear in $[F_{\mu\nu\lambda|\alpha\beta\gamma}]$, as they obviously reduce to total divergences, and thus give a vanishing S_1).

The second type of solutions is associated with $m^\mu \neq 0$, it being understood that we maintain the restriction on the derivative order of a_0 and discard the divergence-like solutions $a_0 = \partial_\mu u^\mu$. Denoting the Euler–Lagrange derivatives of a_0 by $A^{\mu\nu|\alpha\beta} \equiv \delta a_0/\delta t_{\mu\nu|\alpha\beta}$ and using (46), (89) implies that

$$\partial_\mu A^{\mu\nu|\alpha\beta} = 0, \tag{91}$$

where the tensor $A^{\mu\nu|\alpha\beta}$ is imposed to contain at most two derivatives, to have the mixed symmetry of $t_{\mu\nu|\alpha\beta}$ and to fulfill the Bianchi I identity $A^{[\mu\nu|\alpha]\beta} \equiv 0$.

According to the discussion from the end of Sect. 2 (see (33)–(34)), the general solution to (91) is

$$\frac{\delta a_0}{\delta t_{\mu\nu|\alpha\beta}} \equiv A^{\mu\nu|\alpha\beta} = \partial_\rho \partial_\gamma \tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma} + c\left(\sigma^{\mu\alpha}\sigma^{\nu\beta} - \sigma^{\mu\beta}\sigma^{\nu\alpha}\right), \tag{92}$$

where $\tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma}$ has the mixed symmetry of the curvature tensor. The second term in (92) is non-trivial and generates a cosmological-like term

$$a_0^{(1)} = 2ct, \tag{93}$$

where t is the double trace of the tensor field $t_{\mu\nu|\alpha\beta}$. It verifies the equation

$$\gamma a_0^{(1)} = \partial_\mu m^{(1)\mu},\ m^{(1)\mu} = 8c\eta^{\mu\alpha|}{}_{\alpha}, \tag{94}$$

so we can write

$$a_0 = a_0^{(1)} + a_0^{(2)}, \tag{95}$$

with

$$\gamma a_0^{(2)} = \partial_\mu m^{(2)\mu} \tag{96}$$

and

$$\frac{\delta a_0^{(2)}}{\delta t_{\mu\nu|\alpha\beta}} = \partial_\rho \partial_\gamma \tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma}. \tag{97}$$

In the sequel we investigate the form of $a_0^{(2)}$. Imposing that $A^{\mu\nu|\alpha\beta}$ contains at most two derivatives, we find that $\tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma}$ involves only the undifferentiated tensor field $t_{\mu\nu|\alpha\beta}$. Let N be a derivation in the algebra of the fields $t_{\mu\nu|\alpha\beta}$ and of their derivatives that counts the powers of the fields and their derivatives, defined by

$$N = \sum_{n\geq 0} \left(\partial_{\mu_1}\ldots\partial_{\mu_n} t_{\mu\nu|\alpha\beta}\right) \frac{\partial}{\partial\left(\partial_{\mu_1}\ldots\partial_{\mu_n} t_{\mu\nu|\alpha\beta}\right)}. \tag{98}$$

Then, it is easy to see that for every non-integrated density u, we have

$$Nu = t_{\mu\nu|\alpha\beta} \frac{\delta u}{\delta t_{\mu\nu|\alpha\beta}} + \partial_\mu s^\mu, \tag{99}$$

where $\delta u/\delta t_{\mu\nu|\alpha\beta}$ denotes the variational derivative of u. If u is a homogeneous polynomial of order $p > 0$ in the fields and their derivatives, then $Nu = pu$, such that

$$u = \frac{1}{p} t_{\mu\nu|\alpha\beta} \frac{\delta u}{\delta t_{\mu\nu|\alpha\beta}} + \partial_\mu \left(\frac{1}{p} s^\mu\right). \tag{100}$$

As $a_0^{(2)}$ can always be decomposed as a sum of homogeneous polynomials of various orders in the fields and their derivatives, it is enough to analyze (96) for a fixed value of p. Setting $u = a_0^{(2)}$ in (100) and using (97), we find that

$$a_0^{(2)} = \frac{1}{p} t_{\mu\nu|\alpha\beta} \partial_\rho \partial_\gamma \tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma} + \partial_\mu \tilde{s}^\mu. \tag{101}$$

Moving the derivatives from $\tilde{\Phi}$ in (101) and taking into account the mixed symmetry of $\tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma}$, we infer that

$$a_0^{(2)} = k F_{\mu\nu\rho|\alpha\beta\gamma} \tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma} + \partial_\mu l^\mu, \tag{102}$$

with $k = 1/9p$. Acting now with γ on (102), we obtain

$$\gamma a_0^{(2)} = -4k\eta_{\xi\eta|\varepsilon} \partial_\delta \left(F_{\mu\nu\rho|\alpha\beta\gamma} \frac{\partial \tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma}}{\partial t_{\delta\varepsilon|\xi\eta}}\right) + \partial_\mu \bar{l}^\mu, \tag{103}$$

for some $\bar{l}^\mu$. From (103) we observe that $a_0^{(2)}$ satisfies (96) if and only if

$$\partial_\delta \left(F_{\mu\nu\rho|\alpha\beta\gamma} \frac{\partial \tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma}}{\partial t_{\delta\varepsilon|\xi\eta}}\right) = 0. \tag{104}$$

Since the quantity between parentheses in (104) has the same mixed symmetry as the tensor field $t_{\delta\varepsilon|\xi\eta}$, with the help of the relations (33)–(34) we determine that

$$F_{\mu\nu\rho|\alpha\beta\gamma} \frac{\partial \tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma}}{\partial t_{\delta\varepsilon|\xi\eta}} = \partial_\varphi \partial_\theta \psi^{\delta\varepsilon\varphi|\xi\eta\theta}, \tag{105}$$

for some $\psi^{\delta\varepsilon\varphi|\xi\eta\theta}$ with the mixed symmetry of the curvature tensor, which depends only on the undifferentiated tensor field $t_{\mu\nu|\alpha\beta}$. Computing the left-hand side of (105), we arrive at

$$\begin{aligned} F_{\mu\nu\rho|\alpha\beta\gamma} \frac{\partial \tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma}}{\partial t_{\delta\varepsilon|\xi\eta}} &= \partial_\varphi \partial_\theta \left(9 t_{\mu\nu|\alpha\beta} \frac{\partial \tilde{\Phi}^{\mu\nu\varphi|\alpha\beta\theta}}{\partial t_{\delta\varepsilon|\xi\eta}}\right) \\ &-9 \frac{\partial^2 \tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma}}{\partial t_{\delta\varepsilon|\xi\eta} \partial t_{\delta'\varepsilon'|\xi'\eta'}} \\ &\times \left(2\left(\partial_\rho t_{\mu\nu|\alpha\beta}\right)\left(\partial_\gamma t_{\delta'\varepsilon'|\xi'\eta'}\right) + t_{\mu\nu|\alpha\beta} \partial_\rho \partial_\gamma t_{\delta'\varepsilon'|\xi'\eta'}\right) \\ &-9 \frac{\partial^3 \tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma}}{\partial t_{\delta\varepsilon|\xi\eta} \partial t_{\delta'\varepsilon'|\xi'\eta'} \partial t_{\delta''\varepsilon''|\xi''\eta''}} \\ &\times t_{\mu\nu|\alpha\beta} \left(\partial_\rho t_{\delta'\varepsilon'|\xi'\eta'}\right)\left(\partial_\gamma t_{\delta''\varepsilon''|\xi''\eta''}\right). \end{aligned} \tag{106}$$

The right-hand side of (106) can be written in the form of the right-hand side from (105) if and only if

$$9 t_{\mu\nu|\alpha\beta} \frac{\partial \tilde{\Phi}^{\mu\nu\varphi|\alpha\beta\theta}}{\partial t_{\delta\varepsilon|\xi\eta}} = \psi^{\delta\varepsilon\varphi|\xi\eta\theta}, \tag{107}$$

$$\frac{\partial^2 \tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma}}{\partial t_{\delta\varepsilon|\xi\eta} \partial t_{\delta'\varepsilon'|\xi'\eta'}} = 0, \tag{108}$$

$$\frac{\partial^3 \tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma}}{\partial t_{\delta\varepsilon|\xi\eta} \partial t_{\delta'\varepsilon'|\xi'\eta'} \partial t_{\delta''\varepsilon''|\xi''\eta''}} = 0.$$

On the one hand, the requirements (108) restrict $\tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma}$ to be linear in $t_{\mu\nu|\alpha\beta}$ and, on the other hand, we have the condition that $\tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma}$ has the same mixed symmetry as the curvature tensor. These considerations fix $\tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma}$ to be precisely of the type

$$\tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma} = k' \Phi^{\mu\nu\rho|\alpha\beta\gamma}, \tag{109}$$

where k' is a real constant and $\Phi^{\mu\nu\rho|\alpha\beta\gamma}$ is the tensor (36) involved in the functions (35) that yield the free field equations. Meanwhile, (109) fixes the value of p from (101) to $p=2$. By direct computation we deduce that (107) is also satisfied and get that

$$\psi^{\delta\varepsilon\varphi|\xi\eta\theta} = 9k'\Phi^{\delta\varepsilon\varphi|\xi\eta\theta}. \qquad (110)$$

Inserting (109) in (101) for $p=2$, due to (35) we infer that

$$a_0^{(2)} = k' t_{\mu\nu|\alpha\beta} T^{\mu\nu|\alpha\beta} + \partial_\mu l^\mu, \qquad (111)$$

and hence (111) is (up to an irrelevant divergence) proportional to the original Lagrangian. This solution is however trivial in $H^0(s|d)$ since it can be expressed as

$$a_0^{(2)} = sb + \partial_\mu v^\mu,\ \mathrm{gh}(b) = -1,\ \mathrm{gh}(v^\mu) = 0, \qquad (112)$$

where

$$b = 4k'\left(t^{*\mu\nu|\alpha\beta} t_{\mu\nu|\alpha\beta} + \eta^{*\mu\nu|\alpha}\eta_{\mu\nu|\alpha} + C^{*\mu\nu}C_{\mu\nu}\right), \qquad (113)$$

$$v^\mu = \left(l^\mu - 16k' t^{*\mu\nu|\alpha\beta}\eta_{\alpha\beta|\nu} - 12k'\eta^{*\alpha\beta|\mu}C_{\alpha\beta}\right). \qquad (114)$$

Then, in agreement with the discussion from the beginning of this section, the solution (112) can be safely removed from the first-order deformation by replacing it with

$$a_0^{(2)} = 0. \qquad (115)$$

From (93) for $c = 1/2$, using (95), and relying on the results contained in the previous subsections, we conclude that

$$S_1 = \int d^D x\, t \qquad (116)$$

represents the only non-trivial first-order deformation of the solution to the master equation for the tensor $t_{\mu\nu|\alpha\beta}$. Moreover, it is consistent to all orders in the coupling constant. Indeed, as $(S_1, S_1) = 0$, the equation (57) that describes the second-order deformation is satisfied with the choice

$$S_2 = 0, \qquad (117)$$

while the remaining higher-order equations are fulfilled for

$$S_3 = S_4 = \ldots = 0, \qquad (118)$$

and hence there are no non-trivial self-interactions for the tensor field $t_{\mu\nu|\alpha\beta}$.

The main conclusion of this section is that, under the general conditions of smoothness, locality, Lorentz covariance and Poincaré invariance of the deformations, combined with the requirement that the interacting Lagrangian is at most second-order derivative, there are no consistent, non-trivial self-interactions for the massless tensor field with the mixed symmetry of the Riemann tensor. The only piece that can be added to the original Lagrangian is a cosmological-like term, which does not modify the original gauge transformations.

6 Interactions with the Pauli–Fierz theory

We have argued in the previous section that there are no consistent self-interactions that can be added to the free action of the massless tensor field $t_{\mu\nu|\alpha\beta}$. In the sequel we investigate if there exist local, smooth, Lorentz-covariant and Poincaré-invariant, consistent interactions between such a tensor field and a non-interacting massless spin-2 field $h_{\mu\nu}$, described by the Pauli–Fierz action [21]. We maintain the restriction on the maximum derivative order of the interactions being equal to two. The self-interactions of a single massless spin-2 field have been extensively studied in the literature and are known to lead to the Einstein–Hilbert action with a cosmological term. We will mainly focus on the cross-couplings, i.e. on the interactions that mix the fields $t_{\mu\nu|\alpha\beta}$ and $h_{\mu\nu}$, and will not insist on the cohomological construction of the Einstein–Hilbert action with a cosmological term, which can be found in detail in [25].

6.1 Free model and accompanying BRST symmetry

We start from a free action, written as the sum between (1) and the Pauli–Fierz action in $D \geq 5$ spacetime dimensions:

$$S_0[t_{\mu\nu|\alpha\beta}, h_{\mu\nu}] = S_0[t_{\mu\nu|\alpha\beta}] + S_0^{\mathrm{PF}}[h_{\mu\nu}], \qquad (119)$$

with

$$S_0^{\mathrm{PF}}[h_{\mu\nu}] = \int d^D x \left(-\frac{1}{2}(\partial^\rho h^{\mu\nu})(\partial_\rho h_{\mu\nu}) + (\partial_\rho h^{\rho\mu})(\partial^\lambda h_{\lambda\mu}) - (\partial^\rho h)(\partial^\lambda h_{\lambda\rho}) + \frac{1}{2}(\partial^\rho h)(\partial_\rho h)\right), \qquad (120)$$

where $h_{\mu\nu}$ is symmetric and h denotes its trace. The action (120) is invariant under the abelian and irreducible gauge transformations

$$\delta_\epsilon h_{\mu\nu} = \partial_{(\mu}\epsilon_{\nu)}. \qquad (121)$$

The presence of the gauge transformations (121) shows that the functions that define the field equations of the Pauli–Fierz action

$$\frac{\delta S_0^{\mathrm{PF}}}{\delta h^{\mu\nu}} \equiv -2H_{\mu\nu} \approx 0 \qquad (122)$$

are not all independent, but satisfy the Noether identities

$$\partial^\mu H_{\mu\nu} = 0. \qquad (123)$$

In the above, $H_{\mu\nu}$ represents the linearized Einstein tensor

$$H_{\mu\nu} = K_{\mu\nu} - \frac{1}{2}\sigma_{\mu\nu}K,\ H_{\mu\nu} = H_{\nu\mu}, \qquad (124)$$

with $K_{\mu\nu}$ the linearized Ricci tensor and K the linearized scalar curvature, which are defined with the help of the linearized Riemann tensor

$$K_{\mu\nu|\alpha\beta} = -\frac{1}{2}\left(\partial_\mu\partial_\alpha h_{\nu\beta} - \partial_\nu\partial_\alpha h_{\mu\beta} - \partial_\mu\partial_\beta h_{\nu\alpha} + \partial_\nu\partial_\beta h_{\mu\alpha}\right) \qquad (125)$$

via its simple and, respectively, double trace $K_{\mu\nu} = K^{\alpha}{}_{\mu|\alpha\nu}$, $K = K^{\mu}{}_{\mu}$. The linearized Riemann tensor $K_{\mu\nu|\alpha\beta}$ exhibits the same symmetries and satisfies the same identity (2) as the tensor field $t_{\mu\nu|\alpha\beta}$, but in addition fulfills the Bianchi II identity

$$\partial_{[\lambda} K_{\mu\nu]\,|\alpha\beta} \equiv 0. \tag{126}$$

The most general gauge invariant objects that can be constructed from $h_{\mu\nu}$ are the linearized Riemann tensor $K_{\mu\nu|\alpha\beta}$ and its spacetime derivatives. The Pauli–Fierz action alone describes a free gauge theory of Cauchy order equal to two, so the Cauchy order of the theory (119) is equal to three.

The main features of the Pauli–Fierz theory can be understood in an elegant fashion via the generalized differential complex $\Omega_2(\mathcal{M})$ introduced in Sect. 2. An interesting result refers to the generalized cohomology of $\bar{d}$ on $\Omega_2(\mathcal{M})$, where $\mathcal{M}$ has the trivial topology of $\mathbb{R}^D$, combined with the operation of generalized Hodge duality. Let us consider a symmetric, covariant tensor field $\bar{H}^{\mu\nu}$, subject to the equation

$$\partial_{\mu} \bar{H}^{\mu\nu} = 0. \tag{127}$$

Then there exists a tensor $\bar{\Phi}^{\mu\alpha|\nu\beta}$ with mixed symmetry of the linearized Riemann tensor, such that

$$\bar{H}^{\mu\nu} = \partial_{\alpha}\partial_{\beta}\bar{\Phi}^{\mu\alpha|\nu\beta} + c\sigma^{\mu\nu}, \tag{128}$$

with c an arbitrary real constant. The above statement can easily be verified with respect to the linearized Einstein tensor (124), which satisfies the Noether identity (123) and can indeed be written in the form (128) for $c = 0$

$$H^{\mu\nu} = \partial_{\alpha}\partial_{\beta}\Phi^{\mu\alpha|\nu\beta}, \tag{129}$$

where the corresponding $\Phi^{\mu\alpha|\nu\beta}$ reads

$$\Phi^{\mu\alpha|\nu\beta} = \frac{1}{2}\left(-h^{\mu\nu}\sigma^{\alpha\beta} + h^{\alpha\nu}\sigma^{\mu\beta} + h^{\mu\beta}\sigma^{\alpha\nu} \right. \tag{130}$$
$$\left. -h^{\alpha\beta}\sigma^{\mu\nu} + \left(\sigma^{\mu\nu}\sigma^{\alpha\beta} - \sigma^{\alpha\nu}\sigma^{\mu\beta}\right) h\right).$$

The overall BRST complex comprises the BRST generators introduced in Sect. 3 and associated with the theory (1), as well as the Pauli–Fierz field $h_{\mu\nu}$, the fermionic ghost η_{μ} corresponding to the gauge invariances of (120), together with the antifields $h^{*\mu\nu}$ and $\eta^{*\mu}$ from the Pauli–Fierz sector. The BRST differential of the entire free gauge theory splits like in (41), where the actions of γ and δ on the former BRST generators are expressed by (46)–(50), while on the latter ones are defined by

$$\gamma h_{\mu\nu} = \partial_{(\mu}\eta_{\nu)},\ \gamma\eta_{\mu} = 0, \tag{131}$$
$$\gamma h^{*\mu\nu} = 0 = \gamma\eta^{*\mu}, \tag{132}$$
$$\delta h_{\mu\nu} = 0 = \delta\eta_{\mu}, \tag{133}$$
$$\delta h^{*\mu\nu} = 2H_{\mu\nu},\ \delta\eta^{*\mu} = -2\partial_{\nu}h^{*\nu\mu}. \tag{134}$$

The pure ghost number and antighost number of the BRST generators can partially be found in (42)–(45), while for the Pauli–Fierz field/ghost/antifield sector they are given below:

$$\mathrm{pgh}\,(h_{\mu\nu}) = 0,\ \mathrm{pgh}\,(\eta_{\mu}) = 1,$$
$$\mathrm{pgh}\,(h^{*\mu\nu}) = 0 = \mathrm{pgh}\,(\eta^{*\mu}), \tag{135}$$
$$\mathrm{agh}\,(h_{\mu\nu}) = 0 = \mathrm{agh}\,(\eta_{\mu}),\ \mathrm{agh}\,(h^{*\mu\nu}) = 1,$$
$$\mathrm{agh}\,(\eta^{*\mu}) = 2. \tag{136}$$

In agreement with the general line of the antifield-BRST method, the free BRST differential s for the theory (119) is canonically generated in the antibracket ($s\cdot = (\cdot, S)$) by the solution to the master equation $(S, S) = 0$, which in our case has the form

$$S = S^{\mathrm{t}} + S^{\mathrm{h}}, \tag{137}$$

where S^{t} is given by the right-hand side of (52) and

$$S^{\mathrm{h}} = S_0^{\mathrm{PF}}[h_{\mu\nu}] + \int d^D x\, h^{*\mu\nu}\partial_{(\mu}\eta_{\nu)}. \tag{138}$$

6.2 First-order deformations: $H(\gamma)$ and $H(\delta|d)$

In order to determine the solution to the local first-order deformation equation (58), we proceed like in Sect. 5, namely, we expand the non-integrated density according to the antighost number as in (59) and solve the equivalent tower of equations, given by (63), and (61) and (62). It is convenient to split the first-order deformation into

$$a = a^{\mathrm{h-h}} + a^{\mathrm{t-t}} + a^{\mathrm{h-t}}, \tag{139}$$

where $a^{\mathrm{h-h}}$ denotes the part responsible for the self-interactions of the Pauli–Fierz field, $a^{\mathrm{t-t}}$ is related to the deformations of the tensor field $t_{\mu\nu|\alpha\beta}$, and $a^{\mathrm{h-t}}$ signifies the component that describes only the cross-interactions between $h_{\mu\nu}$ and $t_{\mu\nu|\alpha\beta}$. Then, $a^{\mathrm{h-h}}$ is completely known (for a detailed analysis, see for instance [25])

$$a^{\mathrm{h-h}} = a_0^{\mathrm{h-h}} + a_1^{\mathrm{h-h}} + a_2^{\mathrm{h-h}}, \tag{140}$$

where

$$a_2^{\mathrm{h-h}} = \eta^{*\mu}\eta^{\alpha}\partial_{\mu}\eta_{\alpha}, \tag{141}$$
$$a_1^{\mathrm{h-h}} = -h^{*\mu\nu}\eta^{\alpha}\left(\partial_{\mu}h_{\nu\alpha} + \partial_{\nu}h_{\mu\alpha} - \partial_{\alpha}h_{\mu\nu}\right), \tag{142}$$

and $a_0^{\mathrm{h-h}}$ is the cubic vertex of the Einstein–Hilbert Lagrangian plus a cosmological term. The piece $a^{\mathrm{t-t}}$ has been computed in the previous section and is given by the right-hand side of (116). Inserting (139) in (58) and using the fact that the first two components already obey the equations

$$sa^{\mathrm{h-h}} = \partial_{\mu}u^{\mu},\ sa^{\mathrm{t-t}} = \partial_{\mu}v^{\mu}, \tag{143}$$

it follows that only $a^{\mathrm{h-t}}$ is unknown, being subject to the equation

$$sa^{\mathrm{h-t}} = \partial_{\mu}w^{\mu}. \tag{144}$$

If we develop $a^{\mathrm{h-t}}$ according to the antighost number

$$a^{\mathrm{h-t}} = \sum_{k=0}^{I} a_k^{\mathrm{h-t}}, \qquad \mathrm{agh}\left(a_k^{\mathrm{h-t}}\right) = k,$$
$$\mathrm{gh}\left(a_k^{\mathrm{h-t}}\right) = 0, \qquad \varepsilon\left(a_k^{\mathrm{h-t}}\right) = 0 \tag{145}$$

(the expansion (145) can be assumed, like in the previous section, to end at a finite value of the antighost number, once we require that $a_0^{\mathrm{h-t}}$ is local), then (144) is equivalent to the tower of equations[4]

$$\gamma a_I^{\mathrm{h-t}} = 0, \tag{146}$$

$$\delta a_I^{\mathrm{h-t}} + \gamma a_{I-1}^{\mathrm{h-t}} = \partial_\mu \overset{(I-1)^\mu}{w}, \tag{147}$$

$$\delta a_k^{\mathrm{h-t}} + \gamma a_{k-1}^{\mathrm{h-t}} = \partial_\mu \overset{(k-1)^\mu}{w}, \; I-1 \geq k \geq 1, \tag{148}$$

where $(\overset{(k)^\mu}{w})_{k=\overline{0,I}}$ are some local currents, with $\mathrm{agh}(\overset{(k)^\mu}{w}) = k$.

Equation (146) shows that $a_I^{\mathrm{h-t}} \in H(\gamma)$, such that on the one hand its solution is unique up to trivial (γ-exact) contributions, $a_I^{\mathrm{h-t}} \rightarrow a_I^{\mathrm{h-t}} + \gamma b_I^{\mathrm{h-t}}$, and on the other hand every purely γ-exact solution $a_I^{\mathrm{h-t}} = \gamma b_I^{\mathrm{h-t}}$ can be taken to vanish, $a_I^{\mathrm{h-t}} = 0$. In order to infer the general solution to this equation, we initially examine the structure of $H(\gamma)$. To this end, from (48) and (132) we observe that all the antifields

$$\omega^{*\Theta} = \left(t^{*\mu\nu|\alpha\beta}, h^{*\mu\nu}, \eta^{*\mu\nu|\alpha}, \eta^{*\mu}, C^{*\mu\nu}\right), \tag{149}$$

and their spacetime derivatives belong to $H^0(\gamma)$. Meanwhile, the definition (46) and the first relation in the formula (131) yield the most general γ-closed (and obviously non-trivial) objects constructed from the original tensor fields as the curvature tensor (13), the linearized Riemann tensor (125), and their derivatives. Consequently, $H^0(\gamma)$ is spanned by arbitrary polynomials in $\omega^{*\Theta}$, $F_{\mu\nu\lambda|\alpha\beta\gamma}$, $K_{\mu\nu|\alpha\beta}$ and their derivatives. From (131), we observe that the undifferentiated Pauli–Fierz ghosts η_μ and their antisymmetric first-order derivatives $\partial_{[\mu}\eta_{\nu]}$ belong to $H(\gamma)$, while the symmetric part of their first-order derivatives is γ-exact (see the former relation in (131)), and so are all their second- and higher-order derivatives since

$$\partial_\alpha \partial_\beta \eta_\mu = \frac{1}{2}\gamma\left(\partial_{(\alpha} h_{\beta)\mu} - \partial_\mu h_{\alpha\beta}\right). \tag{150}$$

We have shown in Sect. 5 that the other set of pure ghost number one ghosts, related to the tensor field $t_{\mu\nu|\alpha\beta}$, brings no contribution to $H(\gamma)$. In conclusion, the presence of the Pauli–Fierz field enriches the cohomology of γ, which is no longer trivial at odd pure ghost numbers, as it happened in the case of the tensor field $t_{\mu\nu|\alpha\beta}$ alone. Regarding the ghosts of pure ghost number equal to two, we have seen in the previous section that the only combinations in $H(\gamma)$ constructed from them are polynomials in $C_{\mu\nu}$ and $\partial_{[\mu}C_{\nu]\alpha}$. Thus, the general solution to (146) is expressed (up to γ-exact objects) by

$$a_I^{\mathrm{h-t}} = \alpha_I^{\mathrm{h-t}}\left([\omega^{*\Theta}], [F_{\mu\nu\lambda|\alpha\beta\gamma}], [K_{\mu\nu|\alpha\beta}]\right) \times \omega^I\left(\eta_\mu, \partial_{[\mu}\eta_{\nu]}, C_{\mu\nu}, \partial_{[\mu}C_{\nu]\alpha}\right), \tag{151}$$

for $I > 0$, where the γ-invariant coefficients $\alpha_I^{\mathrm{h-t}}$ are subject to the conditions $\mathrm{agh}\left(\alpha_I^{\mathrm{h-t}}\right) = I$ and $\mathrm{pgh}\left(\alpha_I^{\mathrm{h-t}}\right) = 0$, while the symbol ω^I stands for a generic notation of the elements with pure ghost number equal to I of a basis of polynomials in the corresponding ghosts and their antisymmetric first-order derivatives. In addition, every term in $a_I^{\mathrm{h-t}}$ must contain at least one element from each of the two theories in order to provide effective cross-interactions. As they have a bounded number of derivatives, the quantities $\alpha_I^{\mathrm{h-t}}$ are in fact polynomials in the antifields, in the curvature tensor (13), in the linearized Riemann tensor, and in all their derivatives. They represent the most general non-trivial elements from $H(\gamma)$ at pure ghost number zero and will again be called "invariant polynomials" (for the larger free gauge theory (119), subject to the gauge symmetries (3) and (121)).

Substituting the solution (151) into the next equation, namely (147), and taking into account the definitions (46)–(50) and (131)–(134), we obtain the result that a necessary condition for (147) to possess (non-trivial) solutions with respect to $a_{I-1}^{\mathrm{h-t}}$ for $I > 0$ is that the invariant polynomials $\alpha_I^{\mathrm{h-t}}$ appearing in (151) are non-trivial elements from $H_I(\delta|d)$, $\delta\alpha_I^{\mathrm{h-t}} = \partial_\mu k^\mu$. Taking into account the fact that the maximum Cauchy order of the free gauge theory (119) is equal to three, we have [24, 26]

$$H_k(\delta|d) = 0, \; k > 3. \tag{152}$$

Meanwhile, the result remains valid that if the invariant polynomial $\alpha_k^{\mathrm{h-t}}$ is trivial in $H_k(\delta|d)$ for $k \geq 3$, then it can be chosen to be trivial also in $H_k^{\mathrm{inv}}(\delta|d)$[5], which combined with (152) allows us to state that

$$H_k^{\mathrm{inv}}(\delta|d) = 0, \; k > 3, \tag{153}$$

where $H_k^{\mathrm{inv}}(\delta|d)$ denotes, just like before, the local cohomology group of the Koszul–Tate differential at antighost number k in the space of invariant polynomials. On account of the definitions (50) and (134), we are able to identify the non-trivial representatives of $(H_k(\delta|d))_{k\geq 2}$, as well as of $\left(H_k^{\mathrm{inv}}(\delta|d)\right)_{k\geq 2}$, under the form

agh	non − trivial representatives spanning $H_k(\delta\|d)$ and $H_k^{\mathrm{inv}}(\delta\|d)$
$k > 3$	none
$k = 3$	$C^{*\mu\nu}$
$k = 2$	$\eta^{*\mu\nu\|\alpha}, \eta^{*\mu}$

(154)

We will exclude, as we did before, all non-trivial elements from $H(\delta|d)$ and $H^{\mathrm{inv}}(\delta|d)$ at strictly positive antighost numbers that involve the spacetime co-ordinates, as they would result in interactions with broken Poincaré invariance. As for the cohomological group $H_1(\delta|d)$, its determination is a difficult task, but we will solve the deformation equations without explicitly computing it.

[4] The fact that it is possible to replace the equation $\gamma a_I^{\mathrm{h-t}} = \partial_\mu \overset{(I)^\mu}{w}$ with (146) can be done like in the proof of Corollary 3.1 from [27].

[5] The proof can be realized in the same manner as that of Theorem 4.1 from [27], with the precaution to include in an appropriate manner the dependence on the Pauli–Fierz sector.

Like in the case of the tensor field $t_{\mu\nu|\alpha\beta}$ alone, the cohomology groups $H_k(\delta|d)$ and $H_k^{\mathrm{inv}}(\delta|d)$ at strictly positive antighost numbers give us information on the obstructions to remove the antifields from the first-order deformation. As a consequence of the result (153), we can eliminate all the terms with $k>3$ from the expansion (145) by adding to it only trivial pieces and thus work with $I\leq 3$. This can be done in principle like in Sect. 5 from [27], up to the following observations:
(1) the cohomological spaces $\left(H^{2l+1}(\gamma)\right)_{l\geq 0}$ are no longer trivial;
(2) the operator $\bar{D}$ should be extended to the Pauli–Fierz ghost sector like in the Appendix A.1 from [25]. The last representative of (145) is of the type (151), with the corresponding invariant polynomials necessarily non-trivial in $H_I^{\mathrm{inv}}(\delta|d)$ for $I=2,3$, and respectively in $H_1(\delta|d)$ for $I=1$.

6.3 The case $I = 3$

In view of the above considerations we can assume that the expansion (145) stops at antighost number three ($I=3$),

$$a^{\mathrm{h-t}}=a_0^{\mathrm{h-t}}+a_1^{\mathrm{h-t}}+a_2^{\mathrm{h-t}}+a_3^{\mathrm{h-t}}, \tag{155}$$

where $a_3^{\mathrm{h-t}}$ is of the form (151) for $I=3$. At this point we enforce the assumption on the maximum derivative order of the corresponding $a_0^{\mathrm{h-t}}$ to be equal to two. Using the result that the most general representative of $H_3^{\mathrm{inv}}(\delta|d)$ is the undifferentiated antifield $C^{*\alpha\beta}$ (see (154) for $k=3$) and that the elements of pure ghost number three that fulfill the condition on the maximum derivative order are given by

$$\left(\eta_\mu\eta_\nu\eta_\rho,\eta_\mu\eta_\nu\partial_{[\rho}\eta_{\lambda]},C_{\mu\nu}\eta_\rho,C_{\mu\nu}\partial_{[\rho}\eta_{\lambda]},\partial_{[\mu}C_{\nu]\rho}\eta_\lambda\right), \tag{156}$$

we can write down that the general solution to (146) for $I=3$ is like

$$\begin{aligned}a_3^{\mathrm{h-t}}&=C^{*\alpha\beta}\left(f_{1\alpha\beta}^{\mu\nu\rho}\eta_\mu\eta_\nu\eta_\rho+f_{2\alpha\beta}^{\mu\nu\rho\lambda}\eta_\mu\eta_\nu\partial_{[\rho}\eta_{\lambda]}+g_{1\alpha\beta}^{\mu\nu\rho}C_{\mu\nu}\eta_\rho\right.\\&\left.+g_{2\alpha\beta}^{\mu\nu\rho\lambda}C_{\mu\nu}\partial_{[\rho}\eta_{\lambda]}+g_{3\alpha\beta}^{\mu\nu\rho\lambda}\partial_{[\mu}C_{\nu]\rho}\eta_\lambda\right)+\gamma b_3,\end{aligned} \tag{157}$$

where all the coefficients of the type f and g are required to be non-derivative constants. Combining this result with the symmetries of the various coefficients due to the corresponding symmetries of the antifield and of the ghosts, we remain with the following independent possibilities in $D\geq 5$ spacetime dimensions:

$$a_3^{\mathrm{h-t}}=a_3^{(1)\mathrm{h-t}}+a_3^{(2)\mathrm{h-t}}+a_3^{(3)\mathrm{h-t}}, \tag{158}$$

where in $D=5$

$$a_3^{(1)\mathrm{h-t}}=\varepsilon^{\alpha\beta\mu\nu\rho}C_{\alpha\beta}^{*}\left(c_1\eta_\mu\eta_\nu\eta_\rho+d_1C_{\mu\nu}\eta_\rho\right)+\gamma b_3^{(1)}; \tag{159}$$

in $D=6$

$$\begin{aligned}a_3^{(2)\mathrm{h-t}}&=\varepsilon^{\alpha\beta\mu\nu\rho\lambda}C_{\alpha\beta}^{*}\left(c_2\eta_\mu\eta_\nu\partial_{[\rho}\eta_{\lambda]}\right.\\&\left.+d_2C_{\mu\nu}\partial_{[\rho}\eta_{\lambda]}+d_3\partial_{[\mu}C_{\nu]\rho}\eta_\lambda\right)+\gamma b_3^{(2)};\end{aligned} \tag{160}$$

in all $D\geq 5$[6]

$$\begin{aligned}a_3^{(3)\mathrm{h-t}}&=C^{*\alpha\beta}\left(c_3\eta_\alpha\eta^\rho\partial_{[\beta}\eta_{\rho]}+d_4C_\alpha{}^\rho\partial_{[\rho}\eta_{\beta]}\right.\\&\left.+d_5\partial_{[\alpha}C_{\beta]\rho}\eta^\rho+d_6\partial_{[\rho}C_{\alpha]\beta}\eta^\rho\right)+\gamma b_3^{(3)}.\end{aligned} \tag{161}$$

In the above all c_m and d_n are real constants. Obviously, since $a_3^{\mathrm{h-t}}$ is subject to (147) for $I=3$ and the components (159)–(161) are mutually independent, it follows that each of them must separately fulfill such an equation, i.e.,

$$\delta a_3^{(i)\mathrm{h-t}}=-\gamma a_2^{(i)\mathrm{h-t}}+\partial_\mu w^{(i)\mu},\ i=1,2,3. \tag{162}$$

By computing the action of δ on $\left(a_3^{(i)\mathrm{h-t}}\right)_{i=1,2,3}$ and using the definitions (47) and (131), we infer that none of them can be written like in the right-hand side of (162), no matter what $\left(b_3^{(i)}\right)_{i=1,2,3}$ we take in the right-hand side of (159)–(161), such that we must set all the nine constants equal to zero:

$$c_m=0,\ m=1,2,3,\ d_n=0,\ n=1,2,3,4,5,6, \tag{163}$$

and so $a_3^{\mathrm{h-t}}=0$.

6.4 The case $I = 2$

We pass to the next eligible value ($I=2$) and write

$$a^{\mathrm{h-t}}=a_0^{\mathrm{h-t}}+a_1^{\mathrm{h-t}}+a_2^{\mathrm{h-t}}. \tag{164}$$

Repeating the reasoning developed in the above, we see that $a_2^{\mathrm{h-t}}$ is, up to trivial γ-exact contributions, of the form (151) for $I=2$, with the elements of pure ghost number two obeying the assumption on the maximum number of derivatives from the corresponding $a_0^{\mathrm{h-t}}$ being equal to two expressed by

$$\left(\eta_\mu\eta_\nu,\eta_\mu\partial_{[\nu}\eta_{\rho]},C_{\mu\nu},\partial_{[\mu}C_{\nu]\rho}\right). \tag{165}$$

Using the fact that the general representative of $H_2^{\mathrm{inv}}(\delta|d)$ is spanned in this situation by the undifferentiated antifields $\eta^{*\alpha\beta|\gamma}$ and $\eta^{*\alpha}$ (see (154) for $k=2$), to which we add the requirement that $a_2^{\mathrm{h-t}}$ comprises only terms that effectively mix the ghost/antifield sectors of the starting free theories, and combining these with (151), we obtain

$$a_2^{\mathrm{h-t}}=\eta^{*\alpha\beta|\gamma}\left(g_{1\alpha\beta\gamma}^{\mu\nu}\eta_\mu\eta_\nu+g_{2\alpha\beta\gamma}^{\mu\nu\rho}\eta_\mu\partial_{[\nu}\eta_{\rho]}\right) \tag{166}$$

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[6] Another possible term in $a_3^{(3)\mathrm{h-t}}$ would be $d_7C^{*\alpha\beta}\partial_{[\rho}C_{\alpha]}{}^\rho\eta_\beta$, but it is trivial since it can be written like $\gamma\left(-\frac{d_7}{3}C^{*\alpha\beta}\eta_{\alpha\rho|}{}^\rho\eta_\beta\right)$, and thus we have discarded it from $a_3^{\mathrm{h-t}}$ by putting $d_7=0$.

$$+\eta^{*\alpha}\left(g_{3\alpha}^{\mu\nu}C_{\mu\nu}+g_{4\alpha}^{\mu\nu\rho}\partial_{[\mu}C_{\nu]\rho}\right)+\gamma b_2,$$

where the coefficients denoted by g are imposed to be non-derivative constants. Taking into account the identity $\eta^{*[\alpha\beta|\gamma]}\equiv 0$ and the hypothesis that we work only in $D\geq 5$ spacetime dimensions, we arrive at[7]

$$a_2^{\mathrm{h-t}}=\frac{c'}{2}\eta^{*\alpha\beta|\mu}\partial_{[\alpha}\eta_{\beta]}\eta_\mu+\frac{c''}{2}\eta^{*\alpha\beta|}{}_{\beta}\partial_{[\alpha}\eta_{\mu]}\eta^\mu+\gamma b_2. \tag{167}$$

We will analyze these terms separately. The first one leads to non-vanishing components of antighost number one and respectively zero as solutions to the equations

$$\delta a_2'^{\mathrm{h-t}}+\gamma a_1'^{\mathrm{h-t}}=\partial_\mu \overset{(1)}{w'}{}^{\mu},\ \delta a_1'^{\mathrm{h-t}}+\gamma a_0'^{\mathrm{h-t}}=\partial_\mu \overset{(0)}{w'}{}^{\mu}, \tag{168}$$

where we introduce the notation

$$a_2'^{\mathrm{h-t}}=\frac{c'}{2}\eta^{*\alpha\beta|\mu}\partial_{[\alpha}\eta_{\beta]}\eta_\mu. \tag{169}$$

Indeed, straightforward calculations give as output

$$\begin{aligned}a_1'^{\mathrm{h-t}}&\\=\frac{c'}{2}t^{*\mu\nu|\alpha\beta}&\left(\left(\partial_\mu h_{\nu\alpha}-\partial_\nu h_{\mu\alpha}\right)\eta_\beta+\left(\partial_\alpha h_{\beta\mu}-\partial_\beta h_{\alpha\mu}\right)\eta_\nu\right.\\&\left.-\left(\partial_\mu h_{\nu\beta}-\partial_\nu h_{\mu\beta}\right)\eta_\alpha-\left(\partial_\alpha h_{\beta\nu}-\partial_\beta h_{\alpha\nu}\right)\eta_\mu\right),\end{aligned} \tag{170}$$

$$a_0'^{\mathrm{h-t}}=\frac{c'}{8}T^{\mu\nu|\alpha\beta}\left(h_{\mu\alpha}h_{\nu\beta}-h_{\mu\beta}h_{\nu\alpha}\right), \tag{171}$$

where the tensor $T^{\mu\nu|\alpha\beta}$ is given in (7). In consequence, we obtained a possible form of the first-order deformation for the cross-interactions between the Pauli–Fierz theory and the tensor field $t_{\mu\nu|\alpha\beta}$ as follows:

$$a'^{\mathrm{h-t}}=a_0'^{\mathrm{h-t}}+a_1'^{\mathrm{h-t}}+a_2'^{\mathrm{h-t}}, \tag{172}$$

where the quantities in the right-hand side of (172) are expressed by (169)–(171). However, $a'^{\mathrm{h-t}}$ is trivial in the context of the overall non-integrated density $a^{\mathrm{h-t}}$ of the first-order deformation in the sense that it is in a trivial class of the local cohomology of the free BRST differential $H^0(s|d)$. Indeed, one can check that it can be put in a s-exact modulo d form

$$\begin{aligned}a'^{\mathrm{h-t}}=c's&\left(\frac{1}{3}C^{*\mu\nu}\eta_\mu\eta_\nu-\frac{1}{2}\eta^{*\alpha\beta|\mu}\left(h_{\alpha\mu}\eta_\beta-h_{\beta\mu}\eta_\alpha\right)\right.\\&\left.+\frac{1}{2}t^{*\mu\nu|\alpha\beta}\left(h_{\mu\alpha}h_{\nu\beta}-h_{\mu\beta}h_{\nu\alpha}\right)\right)+\partial_\mu l^\mu,\end{aligned} \tag{173}$$

and so it can be eliminated from $a^{\mathrm{h-t}}$ by setting

$$c'=0. \tag{174}$$

[7] The possibility $c'''\eta^{*\alpha}\partial_{[\alpha}C_{\nu]}{}^{\nu}$ was excluded from $a_2^{\mathrm{h-t}}$ as it is trivial, being equal to $\gamma\left(-\frac{c'''}{3}\eta^{*\alpha}\eta_{\alpha\nu|}{}^{\nu}\right)$, such that it can be removed from $a_2^{\mathrm{h-t}}$ by choosing $c'''=0$.

The second piece in (167), which is clearly non-trivial, appears to be more interesting. Indeed, let us fix the trivial (γ-exact) contribution from the right-hand side of (167) to

$$b_2=\frac{c''}{2}\eta^{*\alpha\beta|}{}_{\beta}h_{\alpha\gamma}\eta^\gamma, \tag{175}$$

which is equivalent to starting from

$$a_2''^{\mathrm{h-t}}=c''\eta^{*\alpha\beta|}{}_{\beta}\left(\partial_\alpha\eta_\mu\right)\eta^\mu. \tag{176}$$

Then it yields the component of antighost one as solution to the equation $\delta a_2''^{\mathrm{h-t}}+\gamma a_1''^{\mathrm{h-t}}=\partial_\mu \overset{(1)}{w''}{}^{\mu}$ in the form

$$a_1''^{\mathrm{h-t}}=2c''t^{*\mu\alpha}\left(\partial_\mu h_{\alpha\lambda}+\partial_\alpha h_{\mu\lambda}-\partial_\lambda h_{\mu\alpha}\right)\eta^\lambda, \tag{177}$$

where the notation $t^{*\mu\alpha}$ is explained in (40). Next, we pass to the equation

$$\delta a_1''^{\mathrm{h-t}}+\gamma a_0''^{\mathrm{h-t}}=\partial_\mu \overset{(0)}{w''}{}^{\mu}, \tag{178}$$

where

$$\delta a_1''^{\mathrm{h-t}}=-\frac{c''}{2}T^{\mu\alpha}\left(\partial_\mu h_{\alpha\lambda}+\partial_\alpha h_{\mu\lambda}-\partial_\lambda h_{\mu\alpha}\right)\eta^\lambda, \tag{179}$$

with $T^{\mu\alpha}$ given in (8). In the sequel we will show that there are no solutions to (178). Our procedure goes as follows. Suppose that there exist solutions $a_0''^{\mathrm{h-t}}$ to (178). Using the formula (179), it follows that such an $a_0''^{\mathrm{h-t}}$ must be linear in the tensor field $t_{\mu\nu|\alpha\beta}$, quadratic in the Pauli–Fierz field, and second-order in the derivatives. Integrating by parts in the corresponding functional constructed from $a_0''^{\mathrm{h-t}}$ allows us to move the derivatives such as to act only on the Pauli–Fierz fields, and therefore to work with

$$a_0''^{\mathrm{h-t}}=c''t^{\mu\nu|\alpha\beta}a^{\mathrm{lin}}_{\mu\nu|\alpha\beta}\left(h\partial\partial h,\partial h\partial h\right), \tag{180}$$

where the above notation signifies that $a^{\mathrm{lin}}_{\mu\nu|\alpha\beta}$ is a linear combination of the generic polynomials between parentheses (with the mixed symmetry of the tensor field $t_{\mu\nu|\alpha\beta}$). By direct computation we get

$$\begin{aligned}\gamma a_0''^{\mathrm{h-t}}=\partial^\mu&\left(4c''\eta^{\alpha\beta|\nu}a^{\mathrm{lin}}_{\mu\nu|\alpha\beta}\right)\\&-4c''\eta^{\alpha\beta|\nu}\partial^\mu a^{\mathrm{lin}}_{\mu\nu|\alpha\beta}+c''t^{\mu\nu|\alpha\beta}\gamma a^{\mathrm{lin}}_{\mu\nu|\alpha\beta},\end{aligned} \tag{181}$$

where

$$\gamma a^{\mathrm{lin}}_{\mu\nu|\alpha\beta}=\bar{a}^{\mathrm{lin}}_{\mu\nu|\alpha\beta}\left(h\partial\partial\eta,\partial h\partial\eta,\partial\partial h\partial\eta\right), \tag{182}$$

with η being a generic notation for the Pauli–Fierz ghost η_μ. As $\delta a_1''^{\mathrm{h-t}}$ contains no ghosts from the $t_{\mu\nu|\alpha\beta}$-sector, we require that $\gamma a_0''^{\mathrm{h-t}}$ has the property

$$\partial^\mu a^{\mathrm{lin}}_{\mu\nu|\alpha\beta}\left(h\partial\partial h,\partial h\partial h\right)=0, \tag{183}$$

such that

$$\gamma a_0''^{\mathrm{h-t}}=\partial^\mu\left(4c''\eta^{\alpha\beta|\nu}a^{\mathrm{lin}}_{\mu\nu|\alpha\beta}\right)+c''t^{\mu\nu|\alpha\beta}\gamma a^{\mathrm{lin}}_{\mu\nu|\alpha\beta}. \tag{184}$$

Simple calculations in (179) give

$$\delta a_1''^{\mathrm{h-t}} = \partial_\mu p^\mu + c'' t^{\mu\nu|\alpha\beta} b^{\mathrm{lin}}_{\mu\nu|\alpha\beta}\left(\partial h \partial\partial \eta, \partial\partial h \partial \eta, \eta \partial\partial\partial h\right). \tag{185}$$

Inserting (184) and (185) in (178) and observing that only $b^{\mathrm{lin}}_{\mu\nu|\alpha\beta}$ contains terms that are third-order in the derivatives of the Pauli–Fierz fields, we conclude that the existence of $a_0''^{\mathrm{h-t}}$ is completely dictated by the behavior of $b^{\mathrm{lin}}_{\mu\nu|\alpha\beta}$. More precisely, $a_0''^{\mathrm{h-t}}$ exists if and only if the part of the type $\eta\partial\partial\partial h$ from $b^{\mathrm{lin}}_{\mu\nu|\alpha\beta}$ vanishes identically and/or can be written like the δ-variation of something like $\partial h^* t \eta$. Direct computation produces the part from $b^{\mathrm{lin}}_{\mu\nu|\alpha\beta}$ of order three in the derivatives of the Pauli–Fierz fields in the form

$$\begin{aligned} & b^{\mathrm{lin}}_{\mu\nu|\alpha\beta}\left(\eta\partial\partial\partial h\right) \\ & \sim c''\eta^\lambda \partial_\lambda \left(\sigma_{\beta\nu}\left(\partial_\mu \partial^\rho h_{\rho\alpha} + \partial_\alpha \partial^\rho h_{\rho\mu} - \Box h_{\alpha\mu} - \partial_\alpha \partial_\mu h\right)\right. \\ & \quad -\frac{1}{2}\sigma_{\beta\nu}\sigma_{\alpha\mu}\left(\partial^\rho\partial^\gamma h_{\rho\gamma} - \Box h\right) + \partial_\beta \partial_\nu h_{\alpha\mu} \\ & \quad + (\alpha \longleftrightarrow \beta, \mu \longleftrightarrow \nu) \\ & \quad - (\beta \longleftrightarrow \alpha, \mu \to \mu, \nu \to \nu) \\ & \quad \left. - (\mu \longleftrightarrow \nu, \alpha \to \alpha, \beta \to \beta)\right), \end{aligned} \tag{186}$$

and it neither vanishes identically nor is proportional with $\delta\left(\partial_\lambda h^*_{\alpha\mu}\right)$, as it can be observed from the expression (124) of the functions that define the field equations for the Pauli–Fierz field. The rest of the terms from (186) are obtained from the first ones by making the indicated index changes. In conclusion, we must also take

$$c'' = 0 \tag{187}$$

in (176), so $a_2^{\mathrm{h-t}} = 0$.

6.5 The case $I = 1$

Now, we analyze the next possibility, namely $I = 1$ in (145):

$$a^{\mathrm{h-t}} = a_0^{\mathrm{h-t}} + a_1^{\mathrm{h-t}}, \tag{188}$$

where $a_1^{\mathrm{h-t}}$ must be searched among the non-trivial solutions to the equation $\gamma a_1^{\mathrm{h-t}} = 0$, which are offered by

$$\begin{aligned} & a_1^{\mathrm{h-t}} = \\ & \alpha_1^{\mathrm{h-t}}\left(\left[t^{*\mu\nu|\alpha\beta}\right], \left[h^{*\mu\nu}\right], \left[F_{\mu\nu\lambda|\alpha\beta\gamma}\right], \left[K_{\mu\nu|\alpha\beta}\right]\right) \\ & \quad \times \omega^1\left(\eta_\mu, \partial_{[\mu}\eta_{\nu]}\right), \end{aligned} \tag{189}$$

where the elements of pure ghost number one are

$$\left(\eta_\mu, \partial_{[\mu}\eta_{\nu]}\right). \tag{190}$$

On the one hand, the assumption on the maximum derivative order of the interacting Lagrangian being equal to two prevents the coefficients $\alpha_1^{\mathrm{h-t}}$ to depend on either the curvature tensors or their spacetime derivatives. On the other hand, $a_1^{\mathrm{h-t}}$ can involve only the antifields $t^{*\mu\nu|\alpha\beta}$ and their spacetime derivatives, because otherwise, as ω^1 includes only the Pauli–Fierz ghosts, it would not lead to cross-interactions between the fields $t_{\mu\nu|\alpha\beta}$ and $h_{\mu\nu}$. Moving in addition the derivatives from these antifields such as to act only on the elements (190) from $a_1^{\mathrm{h-t}}$ and relying again on the assumption of the maximum derivative order, we eventually remain with one possibility[8] (up to γ-exact quantities)

$$\begin{aligned} & a_1^{\mathrm{h-t}} \\ & \sim t^{*\mu\nu|\alpha\beta}\left(\sigma_{\mu\alpha}\partial_{[\nu}\eta_{\beta]} - \sigma_{\mu\beta}\partial_{[\nu}\eta_{\alpha]} + \sigma_{\nu\beta}\partial_{[\mu}\eta_{\alpha]}\right. \\ & \qquad \left. - \sigma_{\nu\alpha}\partial_{[\mu}\eta_{\beta]}\right) \\ & = 4t^{*\nu\beta}\partial_{[\nu}\eta_{\beta]} \equiv 0, \end{aligned} \tag{191}$$

which vanishes identically due to the symmetry property in (40) of the simple trace of the antifield $t^{*\mu\nu|\alpha\beta}$.

6.6 The case $I = 0$

As $a_1^{\mathrm{h-t}}$ in (191) vanishes, we remain with one more case, namely where $a^{\mathrm{h-t}}$ reduces to its antighost number zero piece

$$a^{\mathrm{h-t}} = a_0^{\mathrm{h-t}}\left(\left[t_{\mu\nu|\alpha\beta}\right], \left[h_{\mu\nu}\right]\right), \tag{192}$$

which is subject to the equation

$$\gamma a_0^{\mathrm{h-t}} = \partial_\mu w^\mu. \tag{193}$$

As we have discussed in Sect. 5, there are two types of solutions to (193). The first one corresponds to $w^\mu = 0$ and is given by arbitrary polynomials that mix the curvature tensor (13) and its spacetime derivatives with the linearized Riemann tensor (125) and its derivatives, which are however excluded from the condition on the maximum derivative order of $a_0^{\mathrm{h-t}}$ (their derivative order is at least four). The second one is associated with $w^\mu \neq 0$, it being understood that we discard the divergence-like solutions $a_0^{\mathrm{h-t}} = \partial_\mu z^\mu$ and preserve the maximum derivative order restriction. Denoting the Euler–Lagrange derivatives of $a_0^{\mathrm{h-t}}$ by $B^{\mu\nu|\alpha\beta} \equiv \delta a_0^{\mathrm{h-t}}/\delta t_{\mu\nu|\alpha\beta}$ and respectively by $D^{\mu\nu} = \delta a_0^{\mathrm{h-t}}/\delta h_{\mu\nu}$, and using the formula (46) together with the first definition in (131), (193) implies that

$$\partial_\mu B^{\mu\nu|\alpha\beta} = 0,\ \partial_\mu D^{\mu\nu} = 0. \tag{194}$$

The tensors $B^{\mu\nu|\alpha\beta}$ and $D^{\mu\nu}$ are imposed to contain at most two derivatives and to have the mixed symmetry of $t_{\mu\nu|\alpha\beta}$ and respectively of $h_{\mu\nu}$. Meanwhile, they must yield a Lagrangian $a_0^{\mathrm{h-t}}$ that effectively couples the two sorts of fields, so $B^{\mu\nu|\alpha\beta}$ and $D^{\mu\nu}$ effectively depend on $h_{\mu\nu}$ and respectively on $t_{\mu\nu|\alpha\beta}$. According to the considerations

[8] The identity $t^{*[\mu\nu|\alpha]\beta} = 0$ forbids the appearance of solutions proportional to Levi-Civita symbols in any $D \geq 5$ dimension.

from Sect. 2 and Sect. 6.1 (see (33) and (34), and (127) and (128)), the solutions to (194) are of the type[9]

$$\frac{\delta a_0^{\mathrm{h-t}}}{\delta t_{\mu\nu|\alpha\beta}} \equiv B^{\mu\nu|\alpha\beta} = \partial_\rho \partial_\gamma \tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma},$$

$$\frac{\delta a_0^{\mathrm{h-t}}}{\delta h_{\mu\nu}} \equiv D^{\mu\nu} = \partial_\alpha \partial_\beta \tilde{\Phi}^{\mu\alpha|\nu\beta}, \qquad (195)$$

where $\tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma}$ and $\tilde{\Phi}^{\mu\alpha|\nu\beta}$ depend only on the undifferentiated fields $h_{\mu\nu}$ and $t_{\mu\nu|\alpha\beta}$ (otherwise, the corresponding $a_0^{\mathrm{h-t}}$ would be more than second-order in the derivatives), with $\tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma}$ having the mixed symmetry of the curvature tensor $F^{\mu\nu\rho|\alpha\beta\gamma}$ and $\tilde{\Phi}^{\mu\alpha|\nu\beta}$ that of the linearized Riemann tensor. From now on we proceed along the lines employed in the Sect. 5.3. In view of this, we introduce a derivation on the algebra of non-integrated densities depending on $t_{\mu\nu|\alpha\beta}$, $h_{\mu\nu}$ and on their derivatives, that counts the powers of the fields and their derivatives,

$$\bar{N} = \sum_{n\geq 0} \left((\partial_{\mu_1} \ldots \partial_{\mu_n} t_{\mu\nu|\alpha\beta}) \frac{\partial}{\partial (\partial_{\mu_1} \ldots \partial_{\mu_n} t_{\mu\nu|\alpha\beta})} + (\partial_{\mu_1} \ldots \partial_{\mu_n} h_{\mu\nu}) \frac{\partial}{\partial (\partial_{\mu_1} \ldots \partial_{\mu_n} h_{\mu\nu})} \right), \qquad (196)$$

and we observe that the action of $\bar{N}$ on an arbitrary non-integrated density $\bar{u}\left([t_{\mu\nu|\alpha\beta}], [h_{\mu\nu}]\right)$ is

$$\bar{N}\bar{u} = t_{\mu\nu|\alpha\beta} \frac{\delta \bar{u}}{\delta t_{\mu\nu|\alpha\beta}} + h_{\mu\nu} \frac{\delta \bar{u}}{\delta h_{\mu\nu}} + \partial_\mu r^\mu, \qquad (197)$$

where $\delta\bar{u}/\delta t_{\mu\nu|\alpha\beta}$ and $\delta\bar{u}/\delta h_{\mu\nu}$ denote the variational derivatives of $\bar{u}$. In the case where $\bar{u}$ is a homogeneous polynomial of order $p > 0$ in the fields and their derivatives, we have $\bar{N}\bar{u} = p\bar{u}$, and so

$$\bar{u} = \frac{1}{p} \left(t_{\mu\nu|\alpha\beta} \frac{\delta \bar{u}}{\delta t_{\mu\nu|\alpha\beta}} + h_{\mu\nu} \frac{\delta \bar{u}}{\delta h_{\mu\nu}} \right) + \partial_\mu \left(\frac{1}{p} r^\mu \right). \qquad (198)$$

As $a_0^{\mathrm{h-t}}$ can always be decomposed as a sum of homogeneous polynomials of various orders, it is enough to analyze the (193) for a fixed value of p. Putting $\bar{u} = a_0^{\mathrm{h-t}}$ in (198) and inserting (195) in the associated relation, we can write

$$a_0^{\mathrm{h-t}} = \frac{1}{p} \left(t_{\mu\nu|\alpha\beta} \partial_\rho \partial_\gamma \tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma} + h_{\mu\nu} \partial_\alpha \partial_\beta \tilde{\Phi}^{\mu\alpha|\nu\beta} \right) + \partial_\mu \bar{r}^\mu. \qquad (199)$$

Moving the derivatives from $\tilde{\Phi}$ in (199) and recalling the mixed symmetries of $\tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma}$ and $\tilde{\Phi}^{\mu\alpha|\nu\beta}$, we infer that

$$a_0^{\mathrm{h-t}} = k_1 F_{\mu\nu\rho|\alpha\beta\gamma} \tilde{\Phi}^{\mu\nu\rho|\alpha\beta\gamma} + k_2 K_{\mu\alpha|\nu\beta} \tilde{\Phi}^{\mu\alpha|\nu\beta} + \partial_\mu \bar{l}^\mu, \qquad (200)$$

with $k_1 = 1/9p$ and $k_2 = -1/2p$. By computing the action of γ on (200) and following a reasoning similar to that applied between the formulas (103) and (111), we obtain that $p = 2$ and

$$a_0^{\mathrm{h-t}} = k' T^{\mu\alpha} h_{\mu\alpha}. \qquad (201)$$

As the above $a_0^{\mathrm{h-t}}$ vanishes on the stationary surface (6) of the field equations for the tensor $t_{\mu\nu|\alpha\beta}$, it is trivial in $H^0(s|d)$. Indeed, by direct computation we have

$$a_0^{\mathrm{h-t}} = s\left(2k' \left(2t^{*\mu\alpha} h_{\mu\alpha} + \eta^{*\alpha\beta|}{}_{\beta} \eta_\alpha\right)\right) + \partial_\mu \left(-8k' t^{*\mu\alpha} \eta_\alpha\right), \qquad (202)$$

so it can be removed from the first-order deformation by choosing

$$k' = 0. \qquad (203)$$

Putting together the results contained in this section, we can state that $S_1^{\mathrm{h-t}} = 0$ and so

$$S_1 = S_1^{\mathrm{h-h}} + S_1^{\mathrm{t-t}}, \qquad (204)$$

where $S_1^{\mathrm{h-h}}$ is the first-order deformation of the solution to the master equation for the Pauli–Fierz theory and $S_1^{\mathrm{t-t}}$ is given in the right-hand side of (116). The consistency of S_1 at the second order in the coupling constant is governed by (57), where $\left(S_1^{\mathrm{h-h}}, S_1^{\mathrm{t-t}}\right) = 0 = \left(S_1^{\mathrm{t-t}}, S_1^{\mathrm{t-t}}\right)$, and thus we have that $S_2^{\mathrm{t-t}} = 0 = S_2^{\mathrm{h-t}}$, while $S_2^{\mathrm{h-h}}$ is highly nontrivial and is known to describe the quartic vertex of the Einstein–Hilbert action, as well as the second-order contributions to the gauge transformations and to the associated non-abelian gauge algebra. The vanishing of $S_1^{\mathrm{h-t}}$ and $S_2^{\mathrm{h-t}}$ further leads, via the equations that stipulate the higher-order deformation equations, to the result that actually

$$S_k^{\mathrm{h-t}} = 0, \; k \geq 1. \qquad (205)$$

The main conclusion of this section is that, under the general conditions of smoothness, locality, Lorentz covariance and Poincaré invariance of the deformations, combined with the requirement that the interacting Lagrangian is at most second-order derivative, there are no consistent, nontrivial cross-couplings between the Pauli–Fierz field and the massless tensor field with the mixed symmetry of the Riemann tensor. The only pieces that can be added to the action (119) are given by the cosmological term for the tensor $t_{\mu\nu|\alpha\beta}$ and, naturally, by the self-interactions of the Pauli–Fierz field, which produce the Einstein–Hilbert action, invariant under diffeomorphisms.

7 Interactions with matter fields

In the final part of this paper we show that the massless tensor field with the mixed symmetry of the Riemann tensor cannot be coupled in a consistent, non-trivial manner to any matter theory such that the matter fields gain gauge transformations. Indeed, let us consider a generic matter theory

$$S^{\mathrm{matt}}[y^i] = \int d^D x \mathcal{L}\left([y^i]\right), \qquad (206)$$

[9] The solutions involving the constant tensors $B^{\mu\nu|\alpha\beta} \sim \left(\sigma^{\mu\alpha}\sigma^{\nu\beta} - \sigma^{\mu\beta}\sigma^{\nu\alpha}\right)$ and $D^{\mu\nu} \sim \sigma^{\mu\nu}$ give cosmological terms and have already been considered in the above. They are not eligible anyway in the present context, which exclusively focuses on the cross-interactions between the two sorts of fields.

where the fields y^i are assumed to have no non-trivial gauge symmetries. In this situation, the BRST differential for the action written as the sum between (1) and (206) acts on the BRST generators according to (46)–(50) and respectively to

$$\gamma y^i = 0,\ \gamma y_i^* = 0,\ \delta y^i = 0,\ \delta y_i^* = -\frac{\delta^L \mathcal{L}}{\delta y^i}, \tag{207}$$

where

$$\mathrm{pgh}\left(y^i\right) = 0 = \mathrm{pgh}\left(y_i^*\right),\ \mathrm{agh}\left(y^i\right) = 0,\ \mathrm{agh}\left(y_i^*\right) = 1, \tag{208}$$

and y_i^* denote the antifields of the matter fields. The presence of the matter theory simply adds to $H(\gamma)$ discussed in Sect. 5.1 the dependence on y^i, y_i^* and their spacetime derivatives, which lie at pure ghost number zero, $[y^i], [y_i^*] \in H^0(\gamma)$, and therefore we still have $H^{2l+1}(\gamma) = 0$. From (207) it is clear that the cross-interactions between the tensor field $t_{\mu\nu|\alpha\beta}$ and the matter fields y^i at the first order in the coupling constant can be produced just by a first-order deformation of the master equation that stops at antighost number one, $a^{\mathrm{t-matt}} = a_1^{\mathrm{t-matt}} + a_0^{\mathrm{t-matt}}$, where $\gamma a_1^{\mathrm{t-matt}} = 0$. However, as $H^1(\gamma)$ is trivial, this fact implies that $a_1^{\mathrm{t-matt}}$ is trivial and consequently the matter fields cannot gain gauge invariance. We remain with the sole possibility that $a^{\mathrm{t-matt}} = a_0^{\mathrm{t-matt}}$, with $\gamma a_0^{\mathrm{t-matt}} = \partial_\mu q^\mu$, whose solutions, once we add the restriction on the maximum derivative order of the cross-couplings being equal to two, are spanned by polynomials that are simultaneously linear in the curvature tensor (13) and of any order in the undifferentiated matter fields.

8 Conclusion

The general conclusion of this paper is that the powerful reformulation of interactions in gauge theories in terms of the local BRST cohomology reveal that the massless tensor field with the mixed symmetry of the Riemann tensor admits no consistent self-interactions and, in the meantime, cannot be coupled in a consistent, non-trivial manner to the massless spin-two field, described in the free limit by the Pauli–Fierz theory. We also argued that the attempt to couple such a mixed symmetry type tensor to purely matter theories produces no gauge transformations with respect to the matter field sector. Our analysis was constantly based on the assumptions that the resulting deformations are smooth, local, Lorentz-covariant and Poincaré-invariant and on the natural requirement that the maximum derivative order of the interacting Lagrangian is equal to two. It is possible that the relaxation of the last condition yields non-trivial, consistent interactions, at least with the massless spin-two fields, in which case the first-order formulation [13,14] of such a tensor field would probably be a happier starting point.

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References

1. T. Curtright, Phys. Lett. B **165**, 304 (1985); T. Curtright, P.G.O. Freund, Nucl. Phys. B **172**, 413 (1980)
2. C.S. Aulakh, I.G. Koh, S. Ouvry, Phys. Lett. B **173**, 284 (1986)
3. J.M. Labastida, T.R. Morris, Phys. Lett. B **180**, 101 (1986); J.M. Labastida, Nucl. Phys. B **322**, 185 (1989)
4. C. Burdik, A. Pashnev, M. Tsulaia, Mod. Phys. Lett. A **16**, 731 (2001)
5. Yu.M. Zinoviev, hep-th/0211233
6. C.M. Hull, JHEP **0109**, 027 (2001)
7. X. Bekaert, N. Boulanger, Commun. Math. Phys. **245**, 27 (2004)
8. X. Bekaert, N. Boulanger, Class. Quant. Grav. **20**, S417 (2003); Phys. Lett. B **561**, 183 (2003)
9. H. Casini, R. Montemayor, L.F. Urrutia, hep-th/0304228
10. N. Boulanger, S. Cnockaert, M. Henneaux, JHEP **0306**, 060 (2003)
11. H. Casini, R. Montemayor, L.F. Urrutia, Phys. Lett. B **507**, 336 (2001); P. de Medeiros, C. Hull, Commun. Math. Phys. **235**, 255 (2003)
12. X. Bekaert, N. Boulanger, M. Henneaux, Phys. Rev. D **67**, 044010 (2003)
13. Yu.M. Zinoviev, hep-th/0304067
14. Yu.M. Zinoviev, hep-th/0306292
15. A.K. Bengtsson, I. Bengtsson, L. Brink, Nucl. Phys. B **227**, 41 (1983)
16. M.A. Vasiliev, Nucl. Phys. B **616**, 106 (2001); Erratum B **652**, 407 (2003)
17. E. Sezgin, P. Sundell, Nucl. Phys. B **634**, 120 (2002)
18. D. Francia, A. Sagnotti, Phys. Lett. B **543**, 303 (2002)
19. C. Bizdadea, C.C. Ciobirca, E.M. Cioroianu, I. Negru, S.O. Saliu, S.C. Sararu, JHEP **0310**, 019 (2003)
20. G. Barnich, M. Henneaux, Phys. Lett. B **311**, 123 (1993)
21. W. Pauli, M. Fierz, Helv. Phys. Acta **12**, 297 (1939); M. Fierz, W. Pauli, Proc. Roy. Soc. Lond. A **173**, 211 (1939)
22. G. Barnich, M. Henneaux, Mod. Phys. Lett. A **7**, 2703 (1992)
23. M. Dubois-Violette, M. Henneaux, Lett. Math. Phys. **49**, 245 (1999); Commun. Math. Phys. **226**, 393 (2002)
24. G. Barnich, F. Brandt, M. Henneaux, Commun. Math. Phys. **174**, 57 (1995); Phys. Rept. **338**, 439 (2000)
25. N. Boulanger, T. Damour, L. Gualtieri, M. Henneaux, Nucl. Phys. B **597**, 127 (2001)
26. G. Barnich, F. Brandt, M. Henneaux, Commun. Math. Phys. **174**, 93 (1995); G. Barnich, M. Henneaux, Phys. Rev. Lett. **72**, 1588 (1994)
27. C. Bizdadea, C.C. Ciobirca, E.M. Cioroianu, S.O. Saliu, S.C. Sararu, hep-th/0402099
28. M. Henneaux, Commun. Math. Phys. **140**, 1 (1991)